Algebra II

On the factorisation of polynomials – Part II

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2.6 Aspects of factorisation III: The remainder theorem and long division

We are now going to look more generally at the structure of a polynomial $f(x)$, specifically what happens to the the structure of $f(x)$ when it is divided by another polynomial $g(x)$, where the degree of $g(x)$ will be less than the degree of $f(x)$.

Before we start let us look at a numerical example. Consider the number 4200. This can be factorized as

$$\begin{aligned} 4200 &= 2 \times 2100 \\ &= 2 \times 2 \times 1050 \\ &= 2 \times 2 \times 2 \times 525 \\ &= 2 \times 2 \times 2 \times 3 \times 175 \\ &= 2 \times 2 \times 2 \times 3 \times 5 \times 35 \\ &= 2 \times 2 \times 2 \times 3 \times 5 \times 5 \times 7 \end{aligned}$$

which can be written more compactly as $4200 = 2^3 \times 3 \times 5^2 \times 7$. Here we have to stop since we can't factorise any further in terms of positive integers. Notice that all the factors are prime numbers. Therefore we have fully factorised 4200 into its prime factors. The above is given the name of the *prime factorization* of 4200. And there is a theorem of number theory, called the *fundamental theorem of arithmetic*, which says that any positive integer *n* can be factorised into prime factors:

$$n = p_1^{k_1} \times p_2^{k_2} \times p_3^{k_3} \times \ldots \times p_m^{k_m} \,, \qquad (1)$$

where $p_1, p_2, \ldots, p_m$ are prime numbers, and where 1 is usually not considered a prime number. For example, $693 = 3^2 \times 7 \times 11$, $731 = 17 \times 43$, $1323 = 3^3 \times 7^2$.

As another example, suppose we want to find positive integers a, b, c, d, e, f such that

$$200772 = 2^a \times 3^b \times 5^c \times 7^d \times 11^e \times 13^f \,.$$

We would do this by first continually dividing 200772 by 2 until this can no longer be done without getting a remainder. In this case we obtain

$$200772 = 2 \times 100386 = 2 \times 2 \times 50193.$$

Now we continually divide by 3 until this can no longer be done without getting a remainder. In this case we have

$$\begin{aligned}200772 &= 2\times 100386\\ &= 2\times 2\times 50193\\ &= 2\times 2\times 3\times 16731\\ &= 2\times 2\times 3\times 3\times 5577\\ &= 2\times 2\times 3\times 3\times 3\times 1859\end{aligned}$$

The number 1859 no longer divides by 3 so we look to divide by 5. Since since 1859 does not end in a 5 or 0 this number cannot be divided by 5. Also, 1859 cannot be divided by 7. Then, after checking for division by 11 and 13 we end up with the prime factorisation of 200772 as

$$200772 = 2^2\times 3^3\times 11\times 13^2.$$

It is also the case from (1) that, if $k_1, k_2, \dots, k_m$ are even, then n is a perfect square. For example,

$$1764 = 2^2\times 3^2\times 7^2 = 42^2.$$

Similarly if $k_1, k_2, \dots, k_m$ are multiples of 3 then n is a perfect cube. For example,

$$1728 = 2^6\times 3^3 = (2^2)^3\times 3^3 = 12^3.$$

A variation on this last example goes as follows: we want to find the smallest integer n such that $n/2$ is a perfect square and $n/3$ is a perfect cube. Since n is divisible by 2 and 3 we can write

$$n = 2^{k_1}\times 3^{k_2}.$$

Finding the smallest n equates to finding the smallest k_1 and k_2. So we now have

$$\frac{n}{2} = 2^{k_1-1}\times 3^{k_2} \qquad \text{and} \qquad \frac{n}{3} = 2^{k_1}\times 3^{k_2-1}$$

Let us first study each of these quations individidually. We want $n/2$ to be a perfect square, i.e. we want

$$\frac{n}{2} = (2^{\cdots}\times 3^{\cdots})^2 = 2^{2\times\cdots}\times 3^{2\times\cdots} \qquad \text{and} \qquad \frac{n}{3} = (2^{\cdots}\times 3^{\cdots})^3 = 2^{3\times\cdots}\times 3^{3\times\cdots}$$

So, for $n/2$ we need both $k_1 - 1$ and k_2 to be a multiple of 2. This implies that $k_1 = 3$ and $k_2 = 2$. Similarly, for $n/3$ we need both k_1 and $k_2 - 1$ to be multiples of 3. This implies that $k_1 = 3$ and $k_2 = 4$.

The value of k_1 is the same for both equations, but which value of k_2 do we choose? We choose the lowest common multiple formed from the k_2 values. In this case it is simply $k_2 = 4$. Hence

$$n = 2^3 \times 3^4 = 648$$

is the number such that $n/2$ is a perfect square and $n/3$ is a perfect cube.

Similarly if we wanted to find the smallest n such that $n/2$ was a perfect square, $n/3$ was a perfect cube, and $n/5$ was a perfect fifth power, then n would be of the form

$$n = 2^a \times 3^b \times 5^c$$

and we would study

$$\frac{n}{2} = 2^{a-1} \times 3^b \times 5^c\,, \qquad \frac{n}{3} = 2^a \times 3^{b-1} \times 5^c\,, \qquad \frac{n}{5} = 2^a \times 3^b \times 5^{c-1}.$$

So for the first equation $a-1$, b, and c have to be multiples of 2, for the second equation a, $b-1$, and c have to be multiples of 3, and for the third equation a, b, and $c-1$ have to be multiples of 5. By these criteria we have the following lowest values for a, b, c:

For the $n/2$ equation	$a = 3$	$b = 2$	$c = 2$
For the $n/3$ equation	$a = 3$	$b = 4$	$c = 3$
For the $n/5$ equation	$a = 5$	$b = 5$	$c = 6$
Lowest common multiple	$a = 15$	$b = 20$	$c = 6$

Hence $n = 2^{15} \times 3^{20} \times 5^6$.

We have seen that, except for prime numbers, every number can be factorised into prime factors. As for the prime number themselves, they will give a remainder upon factorisation. For example, 1327 is a prime number, and so cannot be fully factorised. On the other hand we can write

$$1327 = 1326 + 1 = 2 \times 3 \times 13 \times 17 + 1,$$

and so it seems that prime numbers can be split into a set of multiples and a remainder.

In this section we will look at exactly the same kind of thing as above but this time using olynomials. In other words, we will look at the basic structure a general polynomial $f(x)$. So, if we have an n^{th} degree polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

we want to factorise it into it most basic "prime" components, as well as study the partial factorisation with remainder of $f(x)$. As such we will study what happens when we factorise $f(x)$ into linear factors, some of which may be repeated (i.e. some of the linear factors may be raised to a power), as well as factorising $f(x)$ in quadratic factors, some of which may be repeated.

2.6.1 Division by linear divisors

We know that diving 7 by 2 we obtain

$$\frac{7}{2} = 3\frac{1}{2}.$$

This can be written as

$$\frac{7}{2} = 3 + \frac{1}{2}.$$

where 3 is the quotient and ½ is the remainder. This process is a division process. Another way of dividing 7 by 2 is by asking a multiplication question: how many factors of 2 go into 7?

$$7 = 2 \times 3 + 1.$$

In this case we have essentially found the maximum integer multiple of 2 we can use on 7, without going beyond 7, as a result of which we have obtained a remainder of 1. If the remainder had been 2 or greater then we could have continued dividing until our remainder became less than 2.

So, in general, division of numbers can be defined in two ways: division represents either i) the number of times the divisor goes into the number being divided, or ii) the number of integer multiples of the divisor, along with any remainder

We can do the same thing with polynomials. We can divide them, for example we can do

$$\frac{x^3 + 2x^2 - x - 2}{x - 2}$$

to get a result, and we can also express this result in factored form as

$$x^3 + 2x^2 - x - 2 = (x-2)Q(x) + R(x)\ ,$$

where $Q(x)$ is the quotient we get from doing the division and $R(x)$ is the remainder, also got from the division.

In this section we will look at a way of doing the division of polynomials as well as writing these answers in factored/remainder form.

We will now look at two algorithms for performing the division of polynomials. The first one we will look at relies on removing terms in powers of x, from a polynomial $f(x)$, *by the use of a suitable multiple*. Hence, to divide $f(x) = x^3 + 2x^2 - x - 2$ by $x - 2$ …

Question	**Answer**	**Result**
… what do I have to multiply with $x-2$ in order to get the x^3 term of $f(x)$ so that I can remove this x^3 term?	$-x^2$	$x^3 + 2x^2 - x - 2 - x^2(x-2) = 4x^2 - x - 2$

We continue this process of finding a suitable multiple, and subtracting, until there are no more terms in x:

Question	**Answer**	**Result**
… what do I have to multiply with $x-2$ in order to get the term $4x^2$ of $4x^2 - x - 2$, so that I can remove $4x^2$?	$-4x$	$4x^2 - x - 2 - 4x(x-2) = 7x - 2$
… what do I have to multiply with $x-2$ in order to get the term $7x$ of $7x - 2$ so that I can remove $7x$?	-7	$7x - 2 - 7(x-2) = 12$

We now stop since there are no more terms in x to remove. Now working in reverse we have $7x - 2 - 7(x-2) = 12$ implies

$$7x - 2 = 12 + 7(x-2).$$

But $7x - 2 = 4x^2 - x - 2 - 4x(x-2)$ therefore

$$4x^2 - x - 2 = 12 + 7(x-2) + 4x(x-2).$$

But $4x^2 - x - 2 = x^3 + 2x^2 - x - 2 - x^2(x-2)$ therefore

$$x^3 + 2x^2 - x - 2 = 12 + 7(x-2) + 4x(x-2) + x^2(x-2).$$

This simplifies to

$$x^3 + 2x^2 - x - 2 = (x-2)(x^2 + 4x + 7) + 12.$$

Therefore, $x^3 + 2x^2 - x - 2$ contains $x^2 + 4x + 7$ "lots" of $x - 2$, with 12 left over. So it looks like a polynomial can be expressed in the form

$$f(x) = (x-c)Q(x) + R\ . \tag{2}$$

where the remainder R is a constant. Although (2) is expressed in factorised form, it came from a process of repeated subtraction, i.e. division. What equation (2) tells us that when $f(x)$ is divided by $(x - c)$ the result of such a division can be expressed in the form of a product involving the divisor, a quotient $Q(x)$ of degree one less than that of $f(x)$, and a constant R. This is fact true for any polynomial of degree n, the proof of which we shall see in section 2.6.3

2.6.2 The remainder theorem

Now, if we substitute $x = 2$ into $f(x) = x^3 + 2x^2 - x - 2$ we obtain $f(2) = 12$. But this is the same answer as the one we got at the end of our factoring process above. This is not a coincidence. In general, if we have an n^{th} degree polynomial $f(x)$ and we divide it by a linear term $x - c$ which is not a factor of $f(x)$ we get a remainder which can be found by subsituting $x = c$ into $f(x)$, i.e.

$$f(c) = R. \tag{3}$$

where R is a constant. This is the *remainder theorem*.

Example 1: If we want to find the remainder when $f(x) = x^3 - 7x^2 + 6x + 1$ is divided by $x - 3$, then we substitute $x = 3$ into $f(x)$, i.e. $f(3) = -17$. Hence we can write

$$x^3 - 7x^2 + 6x + 1 = (x - 3)Q(x) - 17\,,$$

where $Q(x)$ is a quadratic, and can be found by comparing coefficients left and right. We could also have done long division to obtain Q and R in one go.

Example 2: To find the remainder when $f(x) = 2x^3 + 3x^2 - 8x + 3$ is divided by $2x - 1$ we substitute $x = 1/2$ into $f(x)$: $f(1/2) = 0$. What this says is that there is no remainder, and this implies that $2x - 1$ fully divides $f(x)$. Therefore $2x - 1$ is a factor of $f(x)$, and we write

$$2x^3 + 3x^2 - 8x + 3 = (2x - 1)Q(x)\,,$$

where $Q(x)$ is a quadratic.

Example 3: If we know that the polynomial $f(x) = 3x^3 + 2x^2 - bx + a$ is divisible by $x - 1$, and gives a remainder of 10 when divided by $x + 1$, we can find the values of a and b as follows: because $f(x)$ is divisible by $x - 1$ we know that $R = 0$. Hence $f(1) = 5 - b + a = 0$. Similarly, $f(-1) = -1 + b + a = 10$. Solving these two equations for a and b gives $a = 3$, $b = 8$. Hence we can write

$$3x^3 + 2x^2 - 8x + 3 = (x - 1)Q(x)$$

and

$$3x^3 + 2x^2 - 8x + 3 = (x + 1)Q(x) + 10\,.$$

Example 4: Given that $x + 2$ divides $f(x) = 2x^3 + 6x^2 + bx - 5$, we can find the remainder when $f(x)$ is divided by $2x - 1$ by first finding b by evaluating $f(-2) = 3 - 2b = 0$, whence $b = 3/2$, and then evaluating $f(1/2)$ using this value of b. Doing so gives us

$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + 6\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)\left(\frac{1}{2}\right) - 5 = -2\frac{1}{2}\,.$$

Example 5: Given that $2x+1$ divides $f(x)=4x^3-7x-3$, solve $4\cos^3\theta-7\cos\theta-3=0$.

Solution: At first sight the two equations in the question seem unrelated since one is a polynomial in x and the other is a trig equation. But if you look carefully you will see that the trig equation has the same structure and coefficients as $f(x)$. So in solving $f(x)$ we will be solving the trig equation.

Hence, from the given information we can factorise $f(x)$ as follows:

$$4x^3-7x-3=(2x+1)Q(x).$$

We know that $Q(x)$ is a quadratic, so we can write the above as

$$4x^3-7x-3=(2x+1)(2x^2+bx+c)\,.$$

(why can we immediately write the term $2x^2$?). Then by expanding and comparing coefficients (left as an exercise) we obtain $0=2b+2$, $-7=2c+b$, and $-3=c$. From $0=2b+2$ we have $b=-1$, hence

$$4x^3-7x-3=(2x+1)(2x^2-x-3)\,.$$

By using any of the usual methods for factorising the quadratic we end up with

$$4x^3-7x-3=(2x+1)(2x-3)(x+1)\,.$$

To solve $4\cos^3\theta-7\cos\theta-3=0$ notice that this is the same as $4x^3-7x-3$ with $x=\cos\theta$. Hence

$$4\cos^3\theta-7\cos\theta-3=(2\cos\theta+1)(2\cos\theta-3)(\cos\theta+1)\,.$$

Therefore

- $2\cos\theta+1=0$ implies $\cos\theta=-1/2$ which implies $\theta=2\pi/3\pm 2n\pi$ and $\theta=4\pi/3\pm 2n\pi$;
- $2\cos\theta-3=0$ implies $\cos\theta=3/2$ which is not a valid answer since $-1\le\cos\theta\le 1$;
- $\cos\theta+1=0$ implies $\cos\theta=-1$ which implies $\theta=\pm\pi\pm 2n\pi$.

Example 6: When a polynomial is divided by $x+2$ the remainder is -19. When it is divided by $x-1$ the remainder is 2. What is the remainder when the polynomial is divided by $(x+2)(x-1)$

Solution: It seems logical to set up the following:

$$f(x)=(x+2)(x-1)Q(x)+R(x)\,. \qquad \text{(a)}$$

Then we somehow solve for $R(x)$. But we don't have enough information to do this. So, instead, we form equations using the linear divisors of $f(x)$. Hence we have

$$f(x) = (x+2)Q_1(x) - 19 \text{ and } f(x) = (x-1)Q_2(x) + 2. \tag{b}$$

Now, we use algebra to combine these two equations so as to form (a). Adding the two equations in (b) will not give us the product of the two divisors; multiplying the two equations in (b) would give us the product of the two divisors, but we would also end up with $[f(x)]^2$ implying the need to take square roots. This may give us an answer but the algebra looks to complicated, and we would end up with the square root of a polynomial.

Instead we do the following, which leads to simpler algebra and no square roots. First we divide both equations by their respective divisors:

$$\text{i)} \quad \frac{f(x)}{x+2} = Q_1(x) - \frac{19}{x+2} \qquad \text{and} \qquad \text{ii)} \quad \frac{f(x)}{x-1} = Q_2(x) + \frac{2}{x-1}$$

Now we subtract i) from ii) to obtain

$$f(x)\left[\frac{1}{x-1} - \frac{1}{x+2}\right] = Q_2(x) - Q_1(x) + \frac{2}{x-1} - \frac{19}{x-2}.$$

Simplifying the rational fractions, and letting $Q_3 = Q_2 - Q_1$ we end up with

$$f(x)\left(\frac{3}{(x-1)(x+2\,)}\right) = Q_3(x) + \frac{21x-15}{(x-1)(x+2)}. \tag{*}$$

Therefore

$$f(x) = \frac{1}{3}(x-1)(x+2\,)Q_3(x) + 7x - 5\,. \tag{**}$$

Since $Q_3(x)/3$ is just another polynomial we can call this simply $Q(x)$. Hence we have

$$f(x) = (x-1)(x+2\,)Q(x) + 7x - 5\,,$$

and the required remainder is $7x - 5$. Following this line of algebra has allowed us to obtain an expression involving the required product of the two divisors by appropriately manipulating the expressions containing the linear divisors (the appropriate manipulation being the rationalisation of fractions).

We could ask what would happen if we added i) and ii) instead of subtracting them. Ultimately we would find that this would not help us in solving the problem. If we were to do so we would obtain

$$f(x) = \frac{(x-1)(x+2)}{2x+1} Q_3(x) + \frac{23-17x}{2x+1}. \qquad (***)$$

in place of (**). But this does not help since it is not of the form $f(x) = (x+2)(x-1)Q(x) + R(x)$. What we would need to do is to find $Q_3(x)/(2x+1)$ in order to break this down into its quotient and remainder. But we don't know $Q_3(x)$ so we can't do this.

However, we could investigate this as follows: we know that

$$\frac{Q_3(x)}{2x+1} = q(x) + \frac{r(x)}{2x+1}.$$

Hence

$$\begin{aligned} f(x) &= (x-1)(x+2)\left[q(x) + \frac{r(x)}{2x+1}\right] + \frac{23-17x}{2x+1} \\ &= (x-1)(x+2)q(x) + \frac{(x-1)(x+2)r(x)}{2x+1} + \frac{23-17x}{2x+1}. \end{aligned}$$

So we would have to solve

$$\frac{(x-1)(x+2)r(x)}{2x+1} + \frac{23-17x}{2x+1} = 7x-5$$

in order to find the $r(x)$ which would make (***) give the same answer as (**). Clearly subtracting i) from ii) is easier.

Exercise: When a polynomial $P(x)$ is divided by $x-19$ the remainder is 99. When it is divided by $x-99$ the remainder is 19. What is the remainder when $P(x)$ is divided by $(x-19)(x-99)$?

2.6.3 *Proof of the remainder theorem – version 1*

The following proof of (3) is based on "A Proof of the REMAINDER THEOREM", J. O. Parker, *The Mathematics Teacher*, Vol. 66, No. 2 (FEBRUARY 1973), p. 142.

Notice that the remainder theorem (3) is based on the fact that $f(x)$ can be expressed in the form (2), namely that any polynomial $f(x)$ can be expressed as a combination of the product of a linear term and some polynomial $Q(x)$ of degree one less than that of $f(x)$, and a constant term. But how do we know we can do this for any polynomial $f(x)$? We don't yet. So we have to prove it. This we shall do by induction. This is where all the heavy work lies. Then it is an easy step to prove the remainder theorem simply by substituting $x = c$ into $f(x) = (x - c)Q(x) + R$.

<u>*Claim:*</u> Given a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-3} + \cdots + a_2x^2 + a_1x + a_0$ then

$$f(x) = (x - c)Q(x) + R$$

where $c \in \mathbb{R}$, $n \in \mathbb{N}$, and $a_i \in \mathbb{R}$, for $i = 0, 1, \ldots, n$, with $a_n \neq 0$, and where the degree of $Q(x)$ is at most $n - 1$.

<u>*Proof:*</u> Let $P(n)$ be the statement

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = (x - c)Q(x) + R. \qquad [*]$$

1. <u>Base case</u>: Let $n = 1$ in [*]. Then we have

$$f(x) = a_1x + a_0\,, \qquad (a)$$

and

$$f(x) = (x - c)Q(x) + R. \qquad (b)$$

With some algebraic manipulation (a) becomes

$$f(x) = a_1(x - c) + a_1c + a_0 \qquad [[*]]$$

which is of the form $(x - c)Q(x) + R$, and where the degree of $Q(x) = a_1$ is indeed less than that of $f(x)$. Also, since $f(c) = a_1c + a_0$, i.e. a constant, [[*]] is of the form necessary to justify factorising (b). Hence $P(1)$ is true.

2. <u>Inductive assumption – Let $n = k$</u>: Let $P(k)$ be true for some positive integer k where $k \leq n$. Then we have $P(k)$ to be some polynomial $h(x)$ of degree k, having a linear factor $(x - c)$. Let this be

$$h(x) = b_kx^k + b_{k-1}x^{k-1} + \cdots + b_1x + b_0 = (x - c)q_1(x) + r_1.$$

3. Inductive step – Let $n = k + 1$. We want to show that $P(k) \Longrightarrow P(k+1)$. In other word we want to show that if $t(x)$ is any polynomial of degree $k + 1$ then

$$t(x) = (x - c)q_2(x) + r_2. \qquad \{*\}$$

Let $t(x)$ be

$$t(x) = m_{n+1}x^{n+1} + m_n x^n + m_{n-1}x^{n-1} + \cdots + m_1 x + m_0\,. \qquad \{\{*\}\}$$

Our aim is to recover {*} from {{*}}. So, factorising x out of all terms except the constant term we obtain

$$\begin{aligned} t(x) &= x(m_{n+1}x^n + m_n x^{n-1} + m_{n-1}x^{n-2} + \cdots + m_1) + b_0\,, \\ &= x.s(x) + b_0\,. \end{aligned}$$

By assumption, $s(x) = (x - c)q_1(x) + r_1$ hence

$$\begin{aligned} t(x) &= x\big((x - c)q_1(x) + r_1\big) + b_0\,, \\ &= x(x - c)q_1(x) + xr_1 + b_0\,. \end{aligned}$$

We now have a term xr_1. It would be nice if we had a term $-cr_1$ so that we could factorise and have a term $(x - c)r_1$. Let us therefore add and subtract the term cr_1 to the right hand side and factorise. We then obtain

$$\begin{aligned} t(x) &= x(x - c)q_1(x) + xr_1 - cr_1 + cr_1 + b_0\,, \\ &= x(x - c)q_1(x) + (x - c)r_1 + cr_1 + b_0\,, \end{aligned}$$

from which we obtain $t(x) = (x - c)(xq_1(x) + r_1) + cr_1 + b_0$, namely

$$t(x) = (x - c)q_2(x) + r_2\,.$$

where $r_2 = cr_1 + b_0$. Hence $P(k) \Rightarrow P(k+1)$, therefore $P(n)$ is true for all $n \in \mathbb{N}$. ■

Comment: Our aim in step 3. is to use whatever maths necessary on {{*}}, along with the inductive assumption, in order to recover {*}. To do this we first had to factorise an x term out of all terms in {*}, except for the constant term. Doing this gave us a factored term $x.s(x)$ and a constant term b_0. Working to get $s(x)$ allowed us to obtain a polynomial of degree k, at which point we could now use the inductive assumption, which is a statement about a polynomial of degree k. Having used the inductive assumption we then added and subtracted an appropriate term (as we did similarly for the base case).

The term we added/subtracted had to be such that we could obtain a second term $(x-c)$, and the reason we wanted this second $(x-c)$ term was so that we could factorise in terms of $(x-c)$. This then allowed us to recover $t(x)$, a polynomial of degree $k+1$, which is what we wanted to show.

Having proved (2) we can now state and prove the remainder theorem as follows:

> **Remainder theorem**:
> Let $f(x)$ be a polynomial of degree n. When $f(x)$ is divided by $x-c$, where c is a real number, the remainder is $f(c)$.

Proof:
Let $f(x)=(x-c)Q(x)+R$, where $f(x)$ is a polynomial of degree $n\in\mathbb{N}$, $Q(x)$ is a polynomial of degree $n-1$ and R is the (constant) remainder. When $x=c$,

$$f(c)=(c-c)Q(x)+R=R. \qquad \blacksquare$$

Comment: The reason we can now just state the form $(x-c)Q(x)+R$ is because we have proved this to be true for any polynomial of degree n. Now see how easy it is to prove the remainder theorem.

Note that the remainder theorem also applies when the divisor is of the form $ax+b$, where $a,b\in\mathbb{R}$. In this case we have

$$f(x)=(ax+b)Q(x)+R.$$

Letting $x=-b/a$ we obtain

$$f(-b/a)=(0)Q(-b/a)+R=R.$$

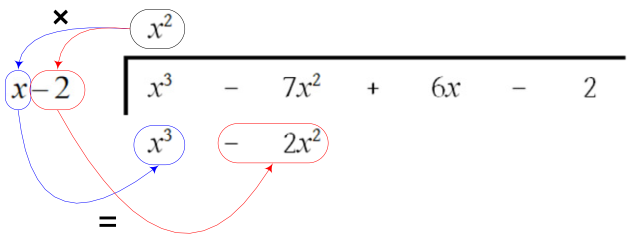
2.6.4 Long division

In the previous section we saw how to perform division by finding suitable multiples by which we could remove terms in powers of x. Another way to perform division is by the method called *long division*. To see how this is done consider a new example, this time of dividing $f(x)=x^3-7x^2+6x-2$ by $g(x)=x-2$.

The set-up for doing this division is

$$
\begin{array}{r|rrrrrrr}
\hline
x-2 & x^3 & - & 7x^2 & + & 6x & - & 2
\end{array}
$$

Now we ask, In our divisor what do we need to multiply x by to get the x^3 of x^3? Answer: x^2. Now we multiply x^2 by the whole of the divisor to get $x^3 - 2x$:



We now subtract $x^3 - 2x$ from $f(x)$ to obtain $-5x^2$, after which we bring down the remaining terms of $f(x)$:

$$
\begin{array}{r|rrrrrrr}
 & x^2 & & & - & 5x & - & 4 \\
\hline
x-2 & x^3 & - & 7x^2 & + & 6x & - & 2 \\
 & x^3 & - & 2x^2 & & & & \\
\cline{2-4}
 & & - & 5x^2 & + & 6x & - & 2
\end{array}
$$

We now repeat the same process on $5x^2 + 6x - 2$, the complete process being shown below:

$$
\begin{array}{r|rrrrrrr}
 & x^2 & & & - & 5x & - & 4 \\
\hline
x-2 & x^3 & - & 7x^2 & + & 6x & - & 2 \\
 & x^3 & - & 2x^2 & & & & \\
\cline{2-4}
 & & - & 5x^2 & + & 6x & - & 2 \\
 & & - & 5x^2 & + & 10x & & \\
\cline{3-6}
 & & & & - & 4x & - & 2 \\
 & & & & - & 4x & + & 8 \\
\cline{5-8}
 & & & & & & - & 10
\end{array}
$$

We can now stop (why?), and what is left is the remainder, and our result can be written as

$$\frac{x^3 - 7x^2 + 6x - 2}{x - 2} = x^2 - 5x - 4 - \frac{10}{x - 2}. \qquad (*)$$

where, here, $x \neq 2$. In the case (*) above notice that

1. the degree of the remainder $R(x)$ is strictly less than the degree of the divisor $g(x)$. In other words, the divisor is linear (an expression of degree 1 in x) implying that the remainder is a constant (an expression of degree 0 in x).

   This will always be the case. If it isn't, $R(x)$ would be at least linear, and we can continue dividing $R(x)$ by g(x). In fact, $\deg\big(R(x)\big) \leq \deg\big(g(x)\big) - 1$.

2. because $f(x)$ is a cubic, and $g(x)$ is linear, then $Q(x)$ is quadratic. We can calculate the difference in degree as $\deg\big(Q(x)\big) = \deg\big(f(x)\big) - \deg\big(f(x)\big)$. This will always be the case. If our divisor had been a quadratic then $Q(x)$ would have been linear.

We can generalise the above result as follows: if we divide a polynomial $f(x)$ of degree n by $(x - c)$, then

$$\frac{f(x)}{x - c} = Q(x) + \frac{R}{x - c}. \qquad (4)$$

where $x \neq c$.

2.6.5 Proof of the remainder theorem – version 2

Having gone through long division we can now prove the remainder theorem is true for all polynomials $f(x)$ of degree n via this method. Therefore let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-3} + \cdots + a_2x^2 + a_1x + a_0\,,$$

where $n \in \mathbb{N}$, and $a_i \in \mathbb{R}$, for $i = 0, 1, \ldots, n$, with $a_n \neq 0$. Let us use long division to divide $f(x)$ by $(x - c)$. The final line of the division is $a_0 + a_1c + a_2c^2 + a_3c^3 + \cdots + a_nc^n$ which is indeed the remainder when $f(x)$ is evaluated at $x = c$.

$$
\begin{array}{r|l}
 & a_n x^{n-1} + x^{n-2}(a_{n-1} + a_n c) + x^{n-3}(a_{n-2} + c a_{n-1} + a_n c^2) \\
\hline
x - c & a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots
\end{array}
$$

$$a_n x^n - a_n c . x^{n-1}$$

$$x^{n-1}(a_{n-1} + a_n c)$$

$$x^{n-1}(a_{n-1} + a_n c) - x^{n-2}(a_{n-1} c + a_n c^2)$$

$$x^{n-2}(a_{n-2} + a_{n-1} c + a_n c^2) -$$

$$x^{n-2}(a_{n-2} + a_{n-1} c + a_n c^2) - x^{n-3}(a_{n-2} c + a_{n-1} c^2 + a_n c^3)$$

$$\ddots \qquad \ddots \qquad \ddots$$

$$\ddots \qquad \ddots \qquad \ddots$$

$$x^{n-k}\left(a_{n-k} + a_{n-(k-1)} c + \cdots + a_n c^k\right)$$

$$x^{n-k}\left(a_{n-k} + a_{n-(k-1)} c + \cdots + a_n c^k\right) - x^{n-(k-1)}\left(a_{n-k} c + a_{n-(k-1)} c^2 + \cdots + a_n c^{k+1}\right)$$

$$x^{n-(k-1)}\left(a_{n-(k-1)} + a_{n-k} c + a_{n-(k-1)} c^2 + \cdots + a_n c^{k+1}\right)$$

$$\ddots \qquad \ddots \qquad \ddots$$

$$\ddots \qquad \ddots \qquad \ddots$$

$$a_0 + a_1 c + a_2 c^2 + a_3 c^3 + \cdots + a_n c^n$$

2.6.6 Some comments about the form of the remainder theorem

Let's have another look at the two forms of expression of the remainder theorem:

$$\frac{f(x)}{x-c} = Q(x) + \frac{R}{x-c} \qquad \text{and} \qquad f(x) = (x-c)Q(x) + R.$$

Form (a) *Form (b)*

I previously mentioned that putting $x = c$ into the latter equation gives us the remainder of $f(x)$ after division by $(x - c)$. But how can we do this with the former equation? We can't since we would be dividing by zero. So it seems that one version of the remainder theorem works at $x = c$ and another version doesn't work at $x = c$. This apparent contradiction can be resolved in any one of three ways.

Approach 1

Here we understand that the division process illustrated by (a) is that of repeated subtraction. In other words we are repeatedly subtracting $(x - c)$ from $f(x)$ until we cannot subtract this anymore. We are then left with a remainder whose degree is less than that of $(x - c)$.

So the division symbol in $f(x)/(x - c)$ simply represents a process of repeated subtraction. Thinking in this manner we see that there is no such thing as division by zero, but rather a stopping of the subtraction process when this cannot be done anymore.

Approach 2

Another way of explaining why (a) is a valid form of the remainder theorem is to do with the idea of limits (a topic usually dealt with in differentiation). The use of limits will then allows us to overcome the problem of $x \neq c$.

So, provided the function $f(x)$ satisfies certain conditions (these being studied during a 1st year maths degree), $f(x)/(x - c)$ will still give us a value if we let x *approach* c. To emphasise the point, as x gets closer and closer to c, forever getting even more close to c, *but not actually equal to* c, there will be an answer to $f(x)/(x - c)$. This is the basic concept of the limiting process.

As an example let us evaluate

$$\frac{x^2 - 1}{x - 1}$$

for various values of x as x approaches the value 1 from just below 1 (i.e. 0.9, 0.99, etc.) and from just above 1 (i.e. 1.1, 1.01, etc).

This is shown in the table below.

x	$(x^2 - 1)/(x - 1)$
0.9	1.9
0.99	1.99
0.999	1.999
0.9999	1.9999
0.99999	1.99999
1	**?**
1.00001	2.00001
1.0001	2.0001
1.001	2.001
1.01	2.01
1.1	2.1

From this table we see that as x approaches 1 from either side the expression $(x^2 - 1)/(x - 1)$ approaches 2 not "∞". In fact, it doesn't matter how close we get to the value 1, the expression $(x^2 - 1)/(x - 1)$ will always approach the value 2.

Using the symbol "→" to mean "approaches" we can say that for form (a)

$$\text{as } x \to c, Q(x) \to 0, \text{ and } \frac{R}{x - c} \to R \text{ and } \frac{f(x)}{x - c} \to f(c)\,.$$

With respect to form (b) we have

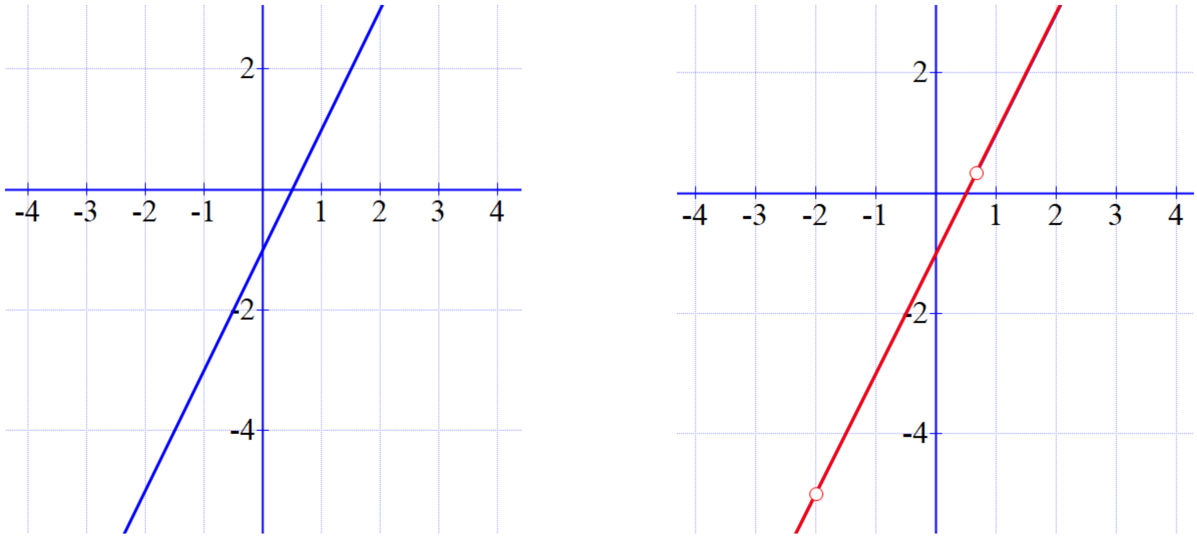
$$\text{as } x \to c, (x - c)Q(x) + R \to R, \text{ and } f(x) \to f(c)\,.$$

Examples

1) Suppose we want to divide $f(x) = 6x^3 + 5x^2 - 12x + 4$ by $g(x) = 3x^2 + 4x - 4$. Then we obtain

$$\frac{6x^3 + 5x^2 - 12x + 4}{3x^2 + 4x - 4} = 2x - 1. \qquad (*)$$

There is no remainder, and the right hand side of (*) is a continuous function as illustrated in the diagram below on the left. The left hand side of (*) is also a continuous function, except that it will have two "holes" to represent the zeroes of the quadratic. One hole will be at $x = -2$ and the other at $x = 2/3$. This is illustrated in the diagram below on the right.

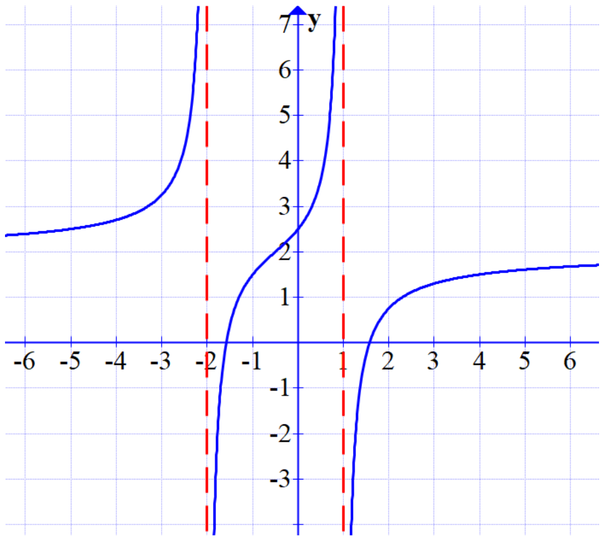


How do we reconcile the fact that the rational function containing holes equates to a function without holes? By realising that $3x^2 + 4x - 4$ completely divides $6x^3 + 5x^2 - 12x + 4$. This means that the denominator cancels out that part of the numerator to leave $2x - 1$ without remainder. This is why we are able to draw the rational function $f(x)/g(x)$ without without holes.

2) Now suppose we want to divide $f(x) = 6x^3 + 5x^2 - 12x + 4$ by $g(x) = 35 - 2x^2 - 9x$. Then we obtain

$$\frac{5 - 2x^2}{2 - x - x^2} = 2(2 - x - x^2) + 2x - 1.$$

The graph of the left hand side is shown below. Here we see that there are two "holes" which are really asymptotes at $x = -2$ and $x = 1$.



From the algebra we see that there is a remainder upon dividing $f(x)$ by $g(x)$, which means that $2 - x - x^2$ does not go completely into $f(x)$. This shows up graphically as the vertical asymptotes in the graph above. Hence, the "holes" in the rational function really are holes, and cannot be "filled in" as in the previous example.

Approach 3:

In modern maths all arithmetic operations start with addition and multiplication. Therefore, every mathematical object tends to be defined in terms of these two operations. For example, we write

$$f(x) = (x - c) \times Q(x) + R\,,$$

instead of

$$\frac{f(x)}{x - c} = Q(x) + \frac{R}{x - c}\,.$$

Therefore, instead of doing division we do multiplication. For example, instead of asking

"what is the result of dividing 45 by 5?"

we ask

"what number do we need to multiply by 5 to get 45?".

As such, we have converted a division problem into an equivalent multiplication problem. And the reason we want to do this is because a number can be multiplied by *absolutely any other number*, including 0, but that same number *cannot* not be divided by absolutely any other number since we cannot be divide by 0 (the same idea applies to subtraction. Subtraction is actually defined as the addition of a negative. So, $5 - 3 = 2$ would formally be written as $5 + (-2) = 3$).

So it is that, because $f(x)/(x-c) = Q(x) + R/(x-c)$ does not have an answer when $x = c$, we convert this to $f(x) = (x-c)Q(x) + R$ which does indeed have an answer when $x = c$.

2.6.7 Synthetic division for linear divisors

The equation $f(x) = (x-a)Q(x) + R$ describes the general structure of a polynomial with linear divisors.. If we want to find the actual quotient and remainder of a polynomial we have to use long division but it is quite an effort to use this. There is a quicker and more efficient way of being able to calculate $Q(x)$ when dividing $f(x)$ by $(x-c)$, and this is called **synthetic division** (synthetic division applies not just for linear divisors but for divisors of any degree. In these notes we will develop synthetic division only for linear and quadratic divisors).

This is how we develop the synthetic division process for linear divisors: let

$$f(x) = a_n x + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 \, , \tag{5}$$

Given $f(x) = (x-c)Q(x) + R$ we know $Q(x)$ is a polynomial of degree one less than $f(x)$ so let us write $Q(x)$ as

$$Q(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + b_{n-3}x^{n-3} + \cdots + b_1 x + b_0 \, , \tag{6}$$

where $b_0, b_1, \ldots, b_{n-3}, b_{n-2}, b_{n-1}$ are coefficients which we need to find. Let us now substitute (6) into the right hand side of $f(x) = (x-c)Q(x) + R$ and expand:

$$\begin{aligned}(x-c)Q(x) + R = \; & b_{n-1}x^{n} + b_{n-2}x^{n-2} + b_{n-3}x^{n-3} + \cdots + b_1 x^2 + b_0 x \\ & -cb_{n-1}x^{n-1} - cb_{n-2}x^{n-2} - cb_{n-3}x^{n-3} - \cdots - cb_1 x + cb_0 + R.\end{aligned} \tag{7}$$

Collecting like terms in powers of x we obtain

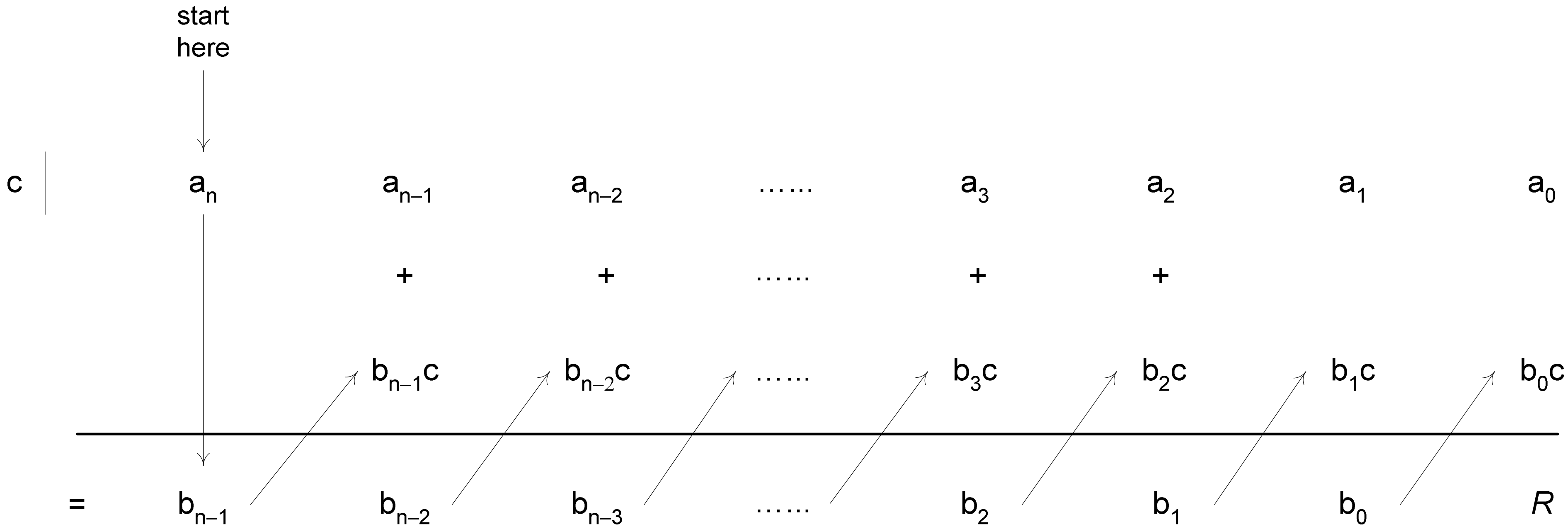
$$\begin{aligned}(x-c)Q(x) + R = \; & b_{n-1}x^{n} + (b_{n-2} - cb_{n-1})x^{n-1} + (b_{n-3} - cb_{n-2})x^{n-2} + \cdots \\ & + \cdots + (b_0 - cb_1)x^2 - cb_0 x + R.\end{aligned} \tag{8}$$

We now compare the right hand side of (8) with the right hand side of (5) to obtain

$$b_{n-1} = a_n \, , \qquad b_{n-2} = a_{n-1} + cb_{n-1} \, , \quad b_{n-3} = a_{n-2} + cb_{n-2} \, , \quad \ldots.,$$

$$\ldots., \quad b_1 = a_2 + cb_2 \, , \qquad b_0 = a_1 + cb_1 \, , \quad R = a_0 + cb_0$$

Arranging this in tabular form we can see more conveniently how the coefficients of $Q(x)$ and R are calculated. This is shown below, and is referred to as *synthetic division*.



Synthetic division of an n^{th} degree polynomial $f(x)$ by a divisor $g(x)$ when $g(x)$ is $(x - c)$, i.e. a linear divisor:

1. The first row consists of the value c which comes from the divisor, and the coefficients of $f(x)$.
2. The second row comes from doing the calculations identified by the diagonal arrows.
3. The third row represent the coefficients of $Q(x)$ and is the sum of the first and the second row.
4. The last value, R, of the third row is the value of the remainder.

You may notice a slight inconsistency in the layout table above. The coefficient a_n relates to x^n of $f(x)$ and the coefficient b_{n-1} relates to x^{n-1} of $Q(x)$, yet we have placed b_{n-1} directly underneath a_n. This can be visually misleading since we might accidentally write $Q(x)$ as $b_{n-1}x^n$.

It is therefore important to remember that the layout of the table is intended as a visual aid designed only to help us keep a track of the coefficients. We have to remember that b_{n-1} relates to x^{n-1}.

<u>*Example:*</u> As an example of the use of synthetic division consider dividing $x^3 - 7x^2 + 6x - 2$ by $(x - 2)$:

1) Form the initial table, with row (1) containing c and the coefficients of $f(x)$:

		(1)	(2)	(3)	(4)
(1)	2	1	–7	6	–2
(2)					
(3)					

2) Bring coefficient a_3 down. This is now b_2 :

		(1)	(2)	(3)	(4)
(1)	2	1	–7	6	–2
(2)					
(3)		1			

3) Multiply c with b_2 and place the answer in row (2) column (2):

		(1)	(2)	(3)	(4)
(1)	2	1	–7	6	–2
(2)			2		
(3)		1			

4) Add the two numbers in column (2). The answer is now b_1 :

		(1)	(2)	(3)	(4)
(1)	2	1	–7	6	–2
(2)			2		
(3)		1	–5		

5) Repeat 3) on column 2, and 4) on column 3, i.e. multiply c by b_1 and place the answer in row (2) column (3):

		(1)	(2)	(3)	(4)
(1)	2	1	–7	6	–2
(2)			2	–10	
(3)		1	–5		

6) Add the two numbers in column (3). The answer is now b_0 :

			(1)	(2)	(3)	(4)
(1)	2		1	–7	6	–2
(2)				2	–10	
(3)			1	–5	–4	

7) Repeat for column (4):

			(1)	(2)	(3)	(4)
(1)	2		1	–7	6	–2
(2)				2	–10	–8
(3)			1	–5	–4	–10

where the last result of -10 is the remainder, and all the numbers before that are the coefficients of $Q(x)$. Therefore

$$x^3 - 7x^2 + 6x - 2 = (x-2)(x^2 - 5x - 4) - 10\,.$$

Another example: Using synthetic division to divide $f(x) = x^3 + 9x^2 + 11x - 21$ by $(x-3)$ we have

3	1	9	11	–21
		3	36	141
	1	12	47	120

where the last result of 120 is the remainder, and all the numbers before that are the coefficients of $Q(x)$. Therefore

$$x^3 + 9x^2 + 11x - 21 = (x - 3)(x^2 + 12x + 47) + 120\,.$$

Another example:

Using synthetic division to divide $f(x) = x^4 + x^3 - 31x^2 + 71x - 42$ by $(x + 7)$ we have

–7	1	1	–31	71	–42
		–7	42	–77	42
	1	–6	11	–6	0

Notice that last result of 0 means that there is no remainder. This implies that $(x + 7)$ divides $f(x)$ completely. Again, all the numbers before the zero are the coefficients of $Q(x)$. Therefore

$$x^4 + x^3 - 31x^2 + 71x - 42 \ = (x + 7)(x^3 - 6x^2 + 11x - 6).$$

2.6.8 Linear divisors of the form $ax - c$

So far our work on division has involved linear divisors of the form $g(x) = x - c$. What if $g(x)$ is of the more general form $g(x) = ax - c$? The usual process of long division will be the same but the synthetic division process will be slightly different (to see why, go back to how the synthetic division coefficients were derived).

There are two ways we can now go. We can either repeat all of the previous analysis which lead us to synthetic division by $x - c$, but this time using $ax - c$, or we can proceed as follows:

- By the remainder theorem we have
$$f(x) = (ax - c)Q(x) + R\ ;$$
- Factorise coefficient a to get
$$f(x) = a\left(x - \frac{c}{a}\right)Q(x) + R\ ;$$
- This is the same as
$$f(x) = \left(x - \frac{c}{a}\right)Q^*(x) + R\ ,$$
where $Q^*(x) = aQ(x)$, and where R remains unchanged.

What this means is that we can perform the standard synthetic division process using the divisor $g^*(x) = (x - c/a)$ to get $Q^*(x)$, and then recover $Q(x)$ by doing $Q^*(x)/a$.

Example

As an example consider using synthetic division to divide $f(x) = 2x^3 - x^2 + 5x + 1$ by $2x - 1$. In this case we will divide $f(x)$ by $x - 1/2$ to give us the quotient $Q^*(x)$, and then divide $Q^*(x)$ by 2 to obtain $Q(x)$:

$$\begin{array}{r|rrrr} 1/2 & 2 & -1 & 5 & 1 \\ & & 1 & 0 & 5/2 \\ \hline & 2 & 0 & 5 & 7/2 \end{array}$$

The quotient above is $Q^*(x) = 2x^2 + 5$ and the remainder is $7/2$. Hence the actual quotient is $Q(x) = x^2 + 5/2$, and the division is given by

$$\frac{2x^3 - x^2 + 5x + 1}{2x - 1} = x^2 + \frac{5}{2} + \frac{7/2}{2x - 1},$$

and the factorisation is given by

$$2x^3 - x^2 + 5x + 1 = (2x - 1)\left(x^2 + \frac{5}{2}\right) + \frac{7}{2}.$$

<u>Another example:</u> Using synthetic division to divide $f(x) = x^4 - 3x^3 + 2x^2 + 5$ by $(3x + 2)$ we first divide by $g^*(x) = x + 2/3$. Hence we obtain

$$
\begin{array}{r|rrrrr}
-\frac{2}{3} & 1 & -3 & 2 & 0 & 5 \\
 & & -\frac{2}{3} & \frac{22}{9} & -\frac{80}{27} & \frac{160}{81} \\
\hline
 & 1 & -\frac{11}{3} & \frac{40}{9} & -\frac{80}{27} & \frac{565}{81}
\end{array}
$$

The quotient above is $Q^*(x) = x^3 - 11x^2/3 + 40x/9 - 80/27$ and the remainder is $565/81$. Hence the actual quotient is $Q(x) = Q^*(x)/3 = x^3/3 - 11x^2/9 + 40x/27 - 80/81$, and the division is given by

$$\frac{x^4 - 3x^3 + 2x^2 + 5}{3x + 2} = \frac{1}{81}(27x^3 - 99x^2 + 120x - 80) + \frac{565/81}{3x + 2},$$

i.e. the factorisation is given by

$$x^4 - 3x^3 + 2x^2 + 5 = \frac{1}{81}(3x + 2)(27x^3 - 99x^2 + 120x - 80) + \frac{565}{81}.$$

2.6.9 A study of the remainder theorem: An iterative application of the remainder theorem

This section is adapted from *Factor and Remainder Theorems: An Appreciation*, Michael Weiss, in *The Mathematics Teacher*, Vol. 110, No. 2 (September 2016), pp. 153-156.

Given

$$f(x) = (x - c)Q(x) + R$$

we have, by the remainder theorem, $f(c) = R$ so the above expression becomes

$$f(x) = (x - c)Q(x) + f(c) \qquad (*)$$

This expression is true for any value of x. For example, given that

$$x^3 - 7x^2 + 6x - 2 = (x - 2)(x^2 - 5x - 4) - 10$$

we can obtain the remainder -10 on the right hand side simply by evaluating the left hand side at $x = 2$. But we can also evaluate the expression above at $x = 0$ to get

$$\text{LHS:} \quad -2 \quad , \qquad \text{RHS:} \quad (-2)(-4) - 10 = -2$$

Let us now rewrite $Q(x)$ in (*) as $Q_1(x)$, the first of many quotients to come. Hence we consider

$$f(x) = f(c) + (x - c)Q_1(x). \qquad (**)$$

Now divide $Q_1(x)$ by $(x - c)$. This gives $Q_1(x) = Q_1(c) + (x - c)Q_2(x)$. Hence (**) becomes

$$f(x) = f(c) + (x - c)Q_1(c) + (x - c)^2 Q_2(x). \qquad (***)$$

We can now divide $Q_2(x)$ by $(x - c)$ to get $Q_2(x) = Q_2(c) + (x - c)Q_3(x)$. Hence (***) becomes

$$f(x) = f(c) + (x - c)Q_1(c) + (x - c)^2 Q_2(c) + (x - c)^3 Q_3(x).$$

We can continue this process of dividing the quotient to get lower degree quotients until our quotient reduces to a constant. This will give the general form

$$f(x) = f(c) + (x - c)Q_1(c) + (x - c)^2 Q_2(c) + (x - c)^3 Q_3(c) + \cdots + R.$$

This means that our original polynomial $f(x)$, which is a polynomial expressed in terms of powers of x, can be expressed as terms involving powers of $(x - c)$. What we have actually done is to expand $f(x)$ in terms of powers of $(x - c)$. Such an expansion is known as Taylor's series, and is more correctly derived and studied in calculus. What a calculus analysis ends up showing is that

$$Q_1(c) = f'(c), \qquad Q_2(c) = f''(c)/2!\,, \quad Q_3(c) = f'''(c)/3!\,,$$

etc. So this example shows us that aspects of algebra lead to calculus, and that the two subjects are not so separate from each other. The disadvantage with the above presentation is that it makes it look as if the expansion is only possible for polynomials when in fact it is possible for other functions also (such as e^x and trig functions).

2.6.10 The remainder theorem for quadratic divisors

Our analysis of the remainder theorem for linear divisors can be extended to quadratic divisor. As such consider finding the remainder of $f(x) = x^3 - 7x^2 + 6x - 2$ when divided by $x^2 - 2x - 3$. There are two ways of doing this.

<u>*First way*</u>

If the divisor can be factorised into $(x - c)(x - d)$ we can divide $f(x)$ by $(x - c)$ to get

$$f(x) = (x - c)Q(x) + R_1.$$

We can then divide $Q(x)$ by $(x - d)$ to obtain

$$Q(x) = (x - d)P(x) + R_2.$$

Substituting this last equation into $f(x)$ we get

$$
\begin{aligned}
f(x) &= (x-c)[(x-d)P(x)+R_2]+R_1\,, \\
&= (x-c)(x-d)P(x)+xR_2-cR_2+R_1\,, \\
&= (x-c)(x-d)P(x)+Ax+B\,.
\end{aligned}
$$

(notice now that the remainder is a linear function instead of a constant). The above process is like dividing 16 by 3 as follows: 16 = 3×5+1, then 5=3×1+2, so that 16=3×(3×1+2)+1. In terms of polynomial division we can say that the general form of the remainder theorem in the case of a quadratic divisor is

$$f(x)=(x-c)(x-d)Q(x)+Ax+B \tag{9}$$

where A and B are constants. By this way we will have divided $f(x)$ by $(x-c)(x-d)$.

<u>*Example:*</u> To divided $f(x)=x^3-7x^2+6x-2$ by x^2-2x-3 we note that the latter expression can be factorised as $(x-3)(x+1)$. So we firstly divide $f(x)$ by $(x-3)$ to get

$$
\begin{array}{r|rrrrrrr}
\multicolumn{1}{r}{} & x^2 & & & - & 4x & - & 6 \\ \cline{2-8}
x-3 & x^3 & - & 7x^2 & + & 6x & - & 2 \\
 & x^3 & - & 3x^2 & & & & \\ \cline{2-4}
 & & - & 4x^2 & + & 6x & - & 2 \\
 & & - & 4x^2 & + & 12x & & \\ \cline{3-6}
 & & & & - & 6x & - & 2 \\
 & & & & - & 6x & + & 18 \\ \cline{5-8}
 & & & & & & - & 20
\end{array}
$$

giving $f(x)=x^3-7x^2+6x-2=(x-3)(x^2-4x-6)-20$ Now we divide (x^2-4x-6) by $(x+1)$ to obtain

$$
\begin{array}{r|rrrrr}
\multicolumn{1}{r}{} & x & & & - & 5 \\ \cline{2-6}
x+1 & x^2 & - & 4x & - & 6 \\
 & x^2 & + & x & & \\ \cline{2-4}
 & & - & 5x & - & 6 \\
 & & - & 5x & - & 5 \\ \cline{3-6}
 & & & & - & 1
\end{array}
$$

Hence $Q(x) = x^2 - 4x - 6 = (x-5)(x+1) - 1$. Substituting this into $f(x)$ we get

$$\begin{aligned} f(x) &= (x-3)[(x-5)(x+1) - 1] - 20\,, \\ &= (x^2 - 2x - 3)(x-5) - x - 17. \end{aligned}$$

In this case the quotient is $(x-5)$ and the remainder is $-x-17$.

<u>*Second way*</u>

Here we simply divide $f(x)$ directly by the quadratic divisor. Doing this we get

$$\begin{array}{r|rrrrrrr} & x & & & & & - & 5 \\ \hline x^2-2x-3 & x^3 & - & 7x^2 & + & 6x & - & 2 \\ & x^3 & - & 2x^2 & - & 3x & & \\ \hline & & - & 5x^2 & + & 9x & - & 2 \\ & & - & 5x^2 & + & 10x & + & 15 \\ \hline & & & & - & x & - & 17 \end{array}$$

where again we get the quotient is $(x-5)$ and the remainder is $-x-17$. Ultimately, our result can be written as

$$\frac{x^3 - 7x^2 + 6x - 2}{x^2 - 2x - 3} = x - 5 - \frac{x+17}{x^2 - 2x - 3},$$

or, since $x^2 - 2x - 3 = (x-3)(x+1)$

$$x^3 - 7x^2 + 6x - 2 = (x-3)(x+1)(x-5) - x - 17. \qquad [*]$$

If we now substitute $x = 3$ into the right-hand side of [*] we obtain $f(3) = -20$, and if we substitute $x = 3$ into the left-hand side of [*] *we also get* $f(3) = -20$. Similarly, if we substitute $x = -1$ into the left- and right-hand side of [*] we obtain $f(-1) = -16$. The function evaluations at the critical values of x therefore allow us to determine the remainder.

For example, if we divide $f(x) = x^3 - 7x^2 + 6x - 2$ by $(x+2)(x-1)$ we have

$$f(-2) = -50 = -2A + B,$$

and $$f(1) = -2 = A + B,$$

from which we get $A = 16$ and $B = -18$, hence $R(x) = 16x - 18$.

The *remainder theorem for quadratic divisors* can now be stated as follows:

Remainder theorem

Let c and d be real numbers. When an n^{th} degree polynomial $f(x)$ is divided by $(x-c)(x-d)$, the remainder is $R(x) = Ax + B$, where A and B are constant, and are found by solving $f(c) = Ac + B$, and $f(d) = Ad + B$.

Recall that in section 2.6.3 we had to prove the remainder theorem with linear divisor to be true for all polynomials $f(x)$. The same applies here. We have to prove that the remainder theorem for quadratic divisors is true for all polynomials $f(x)$. As for the case of linear divisors we can do this by induction (left as an exercise).

Example 1: Find the remainder when $f(x) = 4x^3 - x^2 + 2x - 1$ is divided by $x^2 + x - 6$.

Solution: Let $g(x) = x^2 + x - 6$. We can factorise $g(x)$ to obtain $g(x) = (x+3)(x-2)$. Substituting $x = -3, 2$ into $f(x)$ we obtain

$$f(-3) = -124$$

and

$$f(2) = 31.$$

Since the general form of the remainder upon dividing by a quadratic is $R(x) = Ax + B$ we have

$$-124 = -3A + B$$

and

$$31 = 2A + B$$

from which we have $A = 4$ and $B = -5$. Hence $R(x) = 4x - 5$.

Example 2: Show that $f(x) = 2x^4 + 5x^3 + 3x^2 + x - 2$ is divisible by $x^2 + x + 1$.

Solution: Notice that we can't factorise the quadratic, hence we write $f(x) = (ax^2 + bx + c)Q(x) + R(x)$. Since $f(x)$ is divisible by the given quadratic, we want to show that $R(x) = 0$. We could show this by using long division, but here we will use the remainder theorem. Therefore

$$2x^4 + 5x^3 + 3x^2 + x - 2 = (x^2 + x + 1)(mx^2 + px + q) + Ax + B.$$

Expanding and comparing coefficients gives

$$\begin{aligned} x^4: &\quad 2 = m && \text{(i)} \\ x^3: &\quad 5 = p + m && \text{(ii)} \\ x^2: &\quad 3 = q + p + m && \text{(iii)} \\ x: &\quad 1 = q + p + A && \text{(iv)} \\ \text{constants}: &\quad -2 = q + B && \text{(v)} \end{aligned}$$

From (ii) we get $p = 3$. From (iii) we get $q = -2$. And therefore from (iv) and (v) we get $A = 0$ and $B = 0$. Therefore $R(x) = 0$.

Example 3: Let $f(x) = 2x^3 + ax^2 + bx - 1$. If $f(1) = 4$ and $f(-1) = -8$, what are the values of a and b? Hence find the remainder when $f(x)$ is divided by $x^2 - 1$.

Solution: By the remainder theorem we have $f(x) = R$. Then

$$f(1) = 2 + a + b - 1 = 4,$$

and

$$f(-1) = -2 + a - b - 1 = -8.$$

Solving these two equations for a and b gives $a = -1$ and $b = 4$. Therefore $f(x) = 2x^3 - x^2 + 4x - 1$. We can now find the remainder of $f(x)$ when dividing by $x^2 - 1$. Note that $x^2 - 1 = (x-1)(x+1)$ which are divisors we have used above. Then for a quadratic divisor we have $R(x) = Ax + B$, hence

$$f(-1) = -8 = -A + B$$

and

$$f(1) = 4 = A + B.$$

Solving these two equations gives $A = 6$ and $B = -2$. Hence the remainder is $R(x) = 6x - 2$.

Example 4: When a polynomial in x is divided by $x - a$ the remainder is R_1, and when it is divided by $x - b$ the remainder is R_2. Find the remainder when the polynomial is divided by $(x-a)(x-b)$.

Solution: Let $f(x)$ be a polynomial in x. When the divisor is $x-a$ we have $f(a)=R_1$, and when the divisor is $x-b$ the remainder is $f(b)=R_2$. Now, for a quadratic divisor $(x-a)(x-b)$ we have $f(x)=(x-a)(x-b)Q(x)+Ax+B$. When $x=a$ we obtain

$$f(a)=Aa+B\ , \qquad \text{(i)}$$

and when the divisor is $x-b$ the remainder is

$$f(b)=Ab+B. \qquad \text{(ii)}$$

But we know that $f(a)=R_1$ and $f(b)=R_2$ hence $R_1=Aa+B$ and $R_2=Ab+B$. Subtracting these two equations gives

$$R_1-R_2=f(a)-f(b)=A(a-b). \qquad \text{(iii)}$$

Similarly, subtracting $a.f(b)$ from $b.f(a)$ gives

$$bR_1-aR_2=b.f(a)-a.f(b)=B(b-a). \qquad \text{(iv)}$$

Solving (iii) and iv) for A and B gives

$$A=\frac{R_1-R_2}{a-b} \qquad \text{and} \qquad B=\frac{bR_1-aR_2}{b-a}$$

So the remainder of $f(x)$ when this is divided by $(x-a)(x-b)$ is

$$\begin{aligned} R(x)=Ax+B &= \frac{R_1-R_2}{a-b}x+\frac{bR_1-aR_2}{b-a} && (*) \\ &= \frac{1}{a-b}[R_1(1-b)+R_2(a-1)]. && (**) \end{aligned}$$

Note that instead proceeding with our algebra to obtain (iv) we could have added (i) and (ii) to obtain

$$R_1+R_2=A(a+b)+2B.$$

We could then add this to (iii) to obtain

$$2R_1=A(a-b)+A(a+b)+2B.$$

But this algebra does not help to elimintae A and B. Hence the direction of algebra performed earlier.

Note an interesting thing: If the two remainders are the same, i.e. if when $f(x)$ is divided by $(x-a)$ and $(x-b)$ separately, the remainder R is the same then (**) reduces to $R(x) = R$, a constant. So we obtain the same remainder when we divided by the quadratic divisor as when we divide by the linear divisors separately. Hence for distinct divisors giving the same remainder any polynomial formed using product of those divisors will also give that same remainder: when $f(a) = f(b) = R$ then $f(x) = (x-a)(x-b)Q(x) + R$.

<u>*Example 5:*</u> Let $f(x)$ be an n^{th} degree polynomial with integer coefficients, and let $f(a) = f(b) = -18$, where a, b are distinct integers, when $f(x)$ is divided by $(x-2)$ and $(x+3)$. Is it possible to obtain an integer value of x for which $f(x)$ has remainder 12?

<u>*Solution:*</u> The values $f(a) = 12$ and $f(b) = 12$ represent remainders when $f(x)$ is divided by the given divisors. The value 12 also repesents the remainder when $f(x)$ is divided by the quadratic $(x-2)\,(x+3)$ (see example 4 above). So instead of considering two separate equations each with one of the above divisors let us form one equation using the quadratic as a divisor, i.e.

$$f(x) = (x-2)\,(x+3)Q(x) - 18.$$

Let the integer we are looking for be m. Then $f(m) = 12$ and

$$f(m) = (m-2)\,(m+3)Q(m) - 18.$$

Hence $$12 + 18 = 30 = (m-2)\,(m+3)Q(m).$$

So the three factors on the RHS of this last equation have to give 30 whose distinct factors are 2, 3, 5. Note that because m is integer, $m-2$ and $m+3$ are integers, hence $Q(m)$ is integer. Hence the factors 2, 3, 5 can be attributed to any one of $(m-2)$, $(m+3)$, and $Q(m)$. In fact the factors can be made up of combinations of $\pm 2, \pm 3, \pm 5$ some of which are shown below.

30	=	$(m-2)$	$(m+3)$	$Q(m)$
	=	2	3	5
	=	−2	−3	5
	=	3	−2	−5
	=	5	−3	−2
	=	−5	−2	3

Any one suitable combination will $f(x)$ to have a remainder of 12. *Question*: Would the above work if the remainder of $f(x)$ was 24?

<u>*Example 6:*</u> Let $f(x)$ be an n^{th} degree polynomial with integer coefficients, and let $f(a) = f(b) = f(c) = f(d) = 5$, where a, b, c, d are distinct integers, when $f(x)$ is divided by $(x+1)$, $(x-2)$, $(x+3)$, and $(x-4)$. Show that there is no integer value of x for which $f(x)$ has remainder 8.

<u>*Solution:*</u> The values $f(a) = 5, f(b) = 5, f(c) = 5, f(d) = 5$, represent remainders when $f(x)$ is divided by the given divisors. The value 5 also repesents the remainder when $f(x)$ is divided by the quartic $(x+1)(x-2)\ (x+3)(x-4)$ (see example 4 above). So instead of considering four separate equations each with one of the above divisors let us form one equation using the quartic as a divisor, i.e.

$$f(x) = (x+1)(x-2)\ (x+3)(x-4)Q(x) + 5.$$

Let the integer we are looking for be m. Then $f(m) = 8$ and

$$f(m) = (m+1)(m-2)\ (m+3)(m-4)Q(m) + 5.$$

Hence $$8 - 5 = 3 = (m+1)(m-2)(m+3)(m-4)Q(m).$$

So the five factors on the RHS of this last equation have to give 3. But 3 is a prime number whose factors (including signs) are $\pm1, \pm3$. These are the only factors (in various combinations) we can attribute to $(m+1)(m-2)(m+3)(m-4)Q(m)$ in order to give the answer 3. One combination is $(3)(1)(-1)(p)(q)$ where p and q have to be distinct from each other and the other three factors. But the only possible factors for p and q which make $(3)(1)(-1)(p)(q)$ equal to 3 are 1 and 1 or –1 and –1 respectively, which are not distinct. This problem occurs for whatever combination of distinct factors we use, some of which are shown in the table below.

3	=	$(m+1)$	$(m-2)$	$(m+3)$	$(m-4)$	$Q(m)$
	=	3	1	–1	?	?
	=	–3	1	–1	?	?
	=	1	-1	3	?	?

Hence there is no value of m which satisfies $f(m) = 5$ for the four distinct factors given.

Now compare example 6 with example 5, and see what makes example 5 solvable and example 6 unsolvable. Think about the type of numbers given as the remainders, and what is required for the sum of these two remainders in order to answer the two examples above.

Example 7: Given that m and n are positive integers show that

$$x^m(b^n - c^n) + b^m(c^n - x^n) + c^m(x^n - b^n)$$

is divisible by $x^2 - x(b+c) + bc$.

Solution: Notice that $x^2 - x(b+c) + bc$ can be factorised as $(x-b)(x-c)$, so that $f(x)$ is divisible by $(x-b)(x-c)$. This means that $f(x)$ is divisible separately by $(x-b)$ and $(x-c)$, implying that the remainder is zero when dividing by $(x-b)$ and $(x-c)$. Hence, letting $x = b$ we obtain

$$\begin{aligned} f(b) &= b^m(b^n - c^n) + b^m(c^n - b^n) + c^m(b^n - b^n) \\ &= b^m(b^n - c^n) - b^m(b^n - c^n) + 0 = 0 \end{aligned}$$

as required. Similarly for $f(c)$ (left as an exercise).

Example 8: When the polynomial $f(x)$ is divided by $(x-1)(x-2)(x-3)$ the remainder equals

$$a(x-2)(x-3) + b(x-3)(x-1) + c(x-1)(x-2).$$

Express constants a, b, c in terms of $f(1)$, $f(2)$ and $f(3)$.

Solution: We have $f(x) = (x-1)(x-2)(x-3)Q(x) + R(x)$, where $R(x)$ is the remainder given. So

$$f(1) = R(1) = a(-1)(-2),$$

$$f(2) = R(2) = b(-1)(1),$$

and

$$f(3) = R(3) = c(2)(1).$$

Solving these for a, b, c we obtain $a = \frac{1}{2}f(1)$, $b = -f(2)$ and $c = \frac{1}{2}f(3)$. Hence

$$R(x) = \frac{1}{2}f(1)(x-2)(x-3) - f(2)(x-3)(x-1) + \frac{1}{2}f(3)(x-1)(x-2).$$

Example 9: Determine the values of n for which the polynomial

$$f(x) = x^n + (x-1)^n + (2x-1)^n - (3^n + 2^n + 1)$$

is divisible by $g(x) = x^2 - x - 2$.

Solution: If $f(x)$ is divisible by $g(x)$ then the remainder is zero. Note that we can factorise $g(x)$ to be $g(x) = (x+1)(x-2)$. Hence

$$\begin{aligned} f(-1) &= (-1)^n + (-2)^n + (-3)^n - (3^n + 2^n + 1) \\ &= \begin{cases} 0 & \text{when } n \text{ is even} \\ -2(3^n + 2^n + 1) & \text{when } n \text{ is odd} \end{cases} \end{aligned}$$

Also, $f(2) = (2)^n + (1)^n + (3)^n - (3^n + 2^n + 1) = 0$ for all n. Hence only even values of n allow for $f(x)$ to be divisible by $g(x)$.

Example 10: Consider dividing $f(x) = x^{81} + x^{49} + x^{25} + x^9 + x$, by $g(x) = x^3 - x$. We can factorise $g(x)$ as $g(x) = x(x+1)(x-1)$. Our remainder will be of the form

$$R(x) = ax^2 + bx + c\ ,$$

Evaluating $f(x)$ and $R(x)$ at $x = -1,0,1$ we see that need to solve three simultaneous equations given by

$$f(0) = 0 = a(0)^2 + b(0) + c\ ,$$

$$f(1) = 5 = a(1)^2 + b(1) + c\ ,$$

and

$$f(-1) = -5 = a(-1)^2 + b(-1) + c.$$

From this we find that $a = 0, b = 5, c = 0$. Hence the remainder is $R(x) = 5x$.

2.6.11 Synthetic division for quadratic divisors.

Just as there is a synthetic division process for linear divisors so there is a synthetic division process for quadratic divisors. Consider again $f(x) = (x-c)(x-d)Q(x) + Ax + B$. In this case $Q(x)$ is a polynomial of degree two less than that of $f(x)$. So we can write

$$Q(x) = b_{n-2}x^{n-2} + b_{n-3}x^{n-3} + b_{n-4}x^{n-4} + \cdots + b_1 x + b_0\ , \qquad (10)$$

where $b_0, b_1, \dots, b_{n-2}$ are coefficients we need to find.

Substituting (10) into the right hand side of $f(x) = (x-c)(x-d)Q(x) + Ax + B$ gives:

$(x-c)(x-d)Q(x) + Ax + B$

$$= (x-c)(x-d)(b_{n-2}x^{n-2} + b_{n-3}x^{n-3} + b_{n-4}x^{n-4} + \cdots + b_1x + b_0) + Ax + B.$$

Expanding the bracketed terms on the right hand side of this expression and collecting like terms in x gives

$$\begin{array}{llllllllll}
\multicolumn{10}{l}{(x-c)(x-d)Q(x) + Ax + B =} \\
b_{n-2}x^n & + & b_{n-3}x^{n-1} & - & (c+d).\,b_{n-2}x^{n-1} & & & & & \\
 & + & b_{n-4}x^{n-2} & - & (c+d).\,b_{n-3}x^{n-2} & + & cd.\,b_{n-2}x^{n-2} & & & \\
 & + & b_{n-5}x^{n-3} & - & (c+d).\,b_{n-4}x^{n-3} & + & cd.\,b_{n-3}x^{n-3} & & & \\
 & + & \ldots & \ldots & \ldots & \ldots & \ldots & & & \\
 & + & b_1x^3 & - & (c+d).\,b_2x^3 & + & cd.\,b_3x^3 & & & \\
 & + & b_0x^2 & - & (c+d).\,b_1x^2 & + & cd.\,b_2x^2 & & & \\
 & & & - & (c+d).\,b_0x & + & cd.\,b_1x & + & cd.\,b_0 & + \quad Ax + B
\end{array} \tag{11}$$

Factorising and collecting like terms in x we obtain

$$\begin{array}{lll}
\multicolumn{3}{l}{(x-c)(x-d)Q(x) + Ax + B =} \\
b_{n-2}x^n & + & x^{n-1}(b_{n-3} - (c+d)b_{n-2}) \\
 & + & x^{n-2}(b_{n-4} - (c+d)b_{n-3} + cd.\,b_{n-2}) \\
 & + & x^{n-3}(b_{n-5} - (c+d)b_{n-4} + cd.\,b_{n-3}) \\
 & + & \ldots \\
 & + & x^3(b_1 - (c+d)b_2 + cd.\,b_3) \\
 & + & x^2(b_0 - (c+d)b_1 + cd.\,b_2) \\
 & + & x(-(c+d)b_0 + cd.\,b_1 + A) \qquad + \quad cd.\,b_0 + B
\end{array} \tag{12}$$

Since $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$ the right hand side of (12) must equal the right hand side of this last equation. Therefore comparing coefficients allows us to find the coefficients of $Q(x)$ to be:

$b_{n-2} = a_n$, $b_{n-3} = a_{n-1} + (c+d)b_{n-2}$, $b_{n-4} = a_{n-2} + (c+d)b_{n-3} - cd.b_{n-2}$ ....

...., $b_0 = a_2 + (c+d)b_1 - cd.b_2$, $A = a_1 + (c+d)b_0 - cd.b_1$, $B = a_0 - cd.b_0$

Arranging this in tabular form we can more conveniently see how the coefficients of $Q(x)$ and R are calculated.

$$
\begin{array}{c|ccccc}
 & \text{start here} & & & & \\
c,\ d & a_n & a_{n-1} & a_{n-2} & a_{n-3} & \dots \\
 & & + & + & + & \\
 & & (c+d).b_{n-2} & (c+d).b_{n-3} & (c+d).b_{n-4} & \dots \\
 & & & - & - & \\
 & & & (c.d).b_{n-2} & (c.d).b_{n-3} & \dots \\
\hline
= & b_{n-2} & b_{n-3} & b_{n-4} & b_{n-5} & \dots
\end{array}
$$

$$
\begin{array}{c|cccccc}
c,\, d & \cdots & a_3 & a_2 & a_1 & a_0 \\
 & & + & + & + & \\
 & \cdots & (c+d).b_2 & (c+d).b_1 & (c+d).b_0 & \\
 & & - & - & - & - \\
 & \cdots & (c.d).b_3 & (c.d).b_2 & (c.d).b_1 & (c.d).b_0 \\
\hline
= & \cdots & b_1 & b_0 & \underbrace{A \qquad\qquad B}_{R\,=\,Ax\,+\,B} &
\end{array}
$$

A generalised version of synthetic division for dividing a polynomial $f(x)$ of degree n by a polynomial $g(x)$ of degree $m < n$ can be found at https://eprints.soton.ac.uk/168861/1/FLH_article_on_polynomial_division.pdf (retrieved on 11/11/2018).

Examples: The following are three examples of using synthetic division with quadratic divisors

1) Divide $f(x) = 2x^3 + x^2 - 5x + 2$ by $(x-3)(x+1)$

$$
\begin{array}{r|rrrr}
3, -1 & 2 & 1 & -5 & 2 \\
 & & 4 & 10 & \\
 & & & -6 & -15 \\
\hline
 & 2 & 5 & 11 & 17
\end{array}
$$

Hence $2x^3 + x^2 - 5x + 2 = (x-3)(x+1)(2x+5) + 11x + 17$.

2) Divide $f(x) = 2x^3 + x^2 - 5x + 2$ by $(x-1)(x+2)$

$$
\begin{array}{r|rrrr}
1, -2 & 2 & 1 & -5 & 2 \\
 & & -2 & 1 & \\
 & & & -4 & 2 \\
\hline
 & 2 & -1 & 0 & 0
\end{array}
$$

Hence $2x^3 + x^2 - 5x + 2 = (x-1)(x+2)(2x+1)$.

3) Divide $f(x) = 3x^4 + 8x^3 - 6x^2 - 8x + 3$ by $x^2 - 4$. We do this by noticing that $x^2 - 4 = (x-2)(x+2)$. Therefore

$$
\begin{array}{r|rrrrr}
-2, 2 & 3 & 8 & -6 & -8 & 3 \\
 & & 0 & 0 & 0 & \\
 & & & -12 & -32 & -24 \\
\hline
 & 3 & 8 & 6 & 24 & 27
\end{array}
$$

Hence $3x^4 + 8x^3 - 6x^2 - 8x + 3 = (x^2 - 4)(3x^2 + 8x + 6) + 24x + 27$.

2.6.12 A modified synthetic division algorithm for quadratic divisors

The following is adapted from "Using synthetic division by quadratics to find rational roots", M. W. Hutchinson, *The Mathematics Teacher*, vol. 64, no. 4 (April 1971), pp. 349-352.

Although the synthetic division table above is the logical form for division by quadratic divisors it might be seen to be cumbersome in its application. Furthermore, and more importantly, supposing we had the quadratic divisor $3x^2 + x + 5$? This cannot be factorized in terms of real numbers so how do we perform synthetic division? In this section we will develop another, more convenient, implementation of synthetic division for such a case.

Therefore, consider wanting to divide $f(x) = 6x^4 - 31x^3 - 2x^2 + 6x + 1$ by $x^2 + x + 5$. By long division we obtain

	$6x^2$					–	$37x$	+	5
$x^2 + x + 5$	$6x^4$	–	$31x^3$	–	$2x^2$	+	$6x$	+	1
	$6x^4$	+	$6x^3$	+	$30x^2$				
		–	$37x^3$	–	$32x^2$	+	$6x$		
		–	$37x^3$	–	$37x^2$	–	$185x$		
				+	$5x^2$	+	$191x$	+	1
				+	$5x^2$	+	$5x$	+	25
							$186x$	–	24

This table can be reduced to a table of coefficient:

	(1)	(2)	(3)	(4)	(5)	(6)
(1)		6			–37	5
(2)	1 , 1 , 5	6	–31	–2	6	1
(3)		6	6	30		
(4)			–37	–32	6	
(5)			–37	–37	–185	
(6)				5	191	1
(7)				5	5	25
(8)					186	–24

The new, more convenient, synthetic division table relies on changing the sign of certain numbers (from $+$ to $-$ or $-$ to $+$) as well as the fact that certain coefficients in the table above are redundant.

For example the coefficient of x^2 in $g(x)$ is always "1" so this can be deleted without affecting the rest of the long division process. Also, the first number in lines (3), (5), and (7) can be deleted since they are duplicates of the numbers directly above them. Also, the numbers in the quotient row (1) are repeated in row/columns (2)/(2), (4)/(3), and (6)/(4), so they too can be deleted. Other simplifications and changes in sign to the above table are also made, full details of which can be found in M. W. Hutchinson's paper referred to above.

Because of all of this simplification certain rows of the table can be shifted so as to make the pattern of the synthetic division arithmetic more obvious and convenient. Ultimately we can reformat the table as

	(1)	(2)	(3)	(4)	(5)	(6)
(1)	–1	6	–31	–2	6	1
(2)	–5			–30	185	–25
(3)			–6	37	–5	
(4)		6	–37	5	186	–24

Alternative synthetic division table for quadratic divisors

The process of setting up such a table is shown by the arrows in the table below, and can be described as follows:

1. Place the coefficients of x and the constant of the divisor in column (1), in the order shown, and change their sign as shown;
2. Bring the leading coefficient of $f(x)$ down to row (4);

3. Follow the multiplication shown by the blue and pink arrows, and the addition shown by the red arrows:
    a. Multiply the number in row/column (4)/(2) by the number in (1)/(1): 6 × −1, to get the number in (3)/(3);
    b. Multiply the number in row/column (4)/(2) by the number in (1)/(2): 6 × −5, to get the number in (2)/(4);
    c. Add the numbers in column (3) to get the number in (4)/(3);
4. Repeat step 3 for the rest of the table;

			(1)	(2)	(3)	(4)	(5)	(6)
						$f(x)$		
				x^4	x^3	x^2	x	const
(1)		x	−1	6	−31	−2	6	1
	$g(x)$							
(2)		const	−5			−30	185	−25
(3)					−6	37	−5	
(4)				6	−37	5	186	−24
			$Q(x)$:	x^2	x	const	A	B

The first three numbers in row (4) are the coefficients of the quotient, and the last two numbers are the coefficients of the remainder, all of which match the coefficients obtained by long division.

The above example therefore shows that we can now perform synthetic division on a polynomial even when the quadratic divisor cannot be factorised.

Examples: The following are two examples of using synthetic division with quadratic divisors which cannot be factorised.

1) Divide $f(x) = x^3 + 2x^2 - 3x - 1$ by $x^2 - 2x + 5$

				$f(x)$		
			x^3	x^2	x	const
$g(x)$	$-x$	2	1	2	-3	-1
	$-$const	-5			-5	-20
				2	8	
			1	4	0	-21
		$Q(x)$:	x	const	A	B

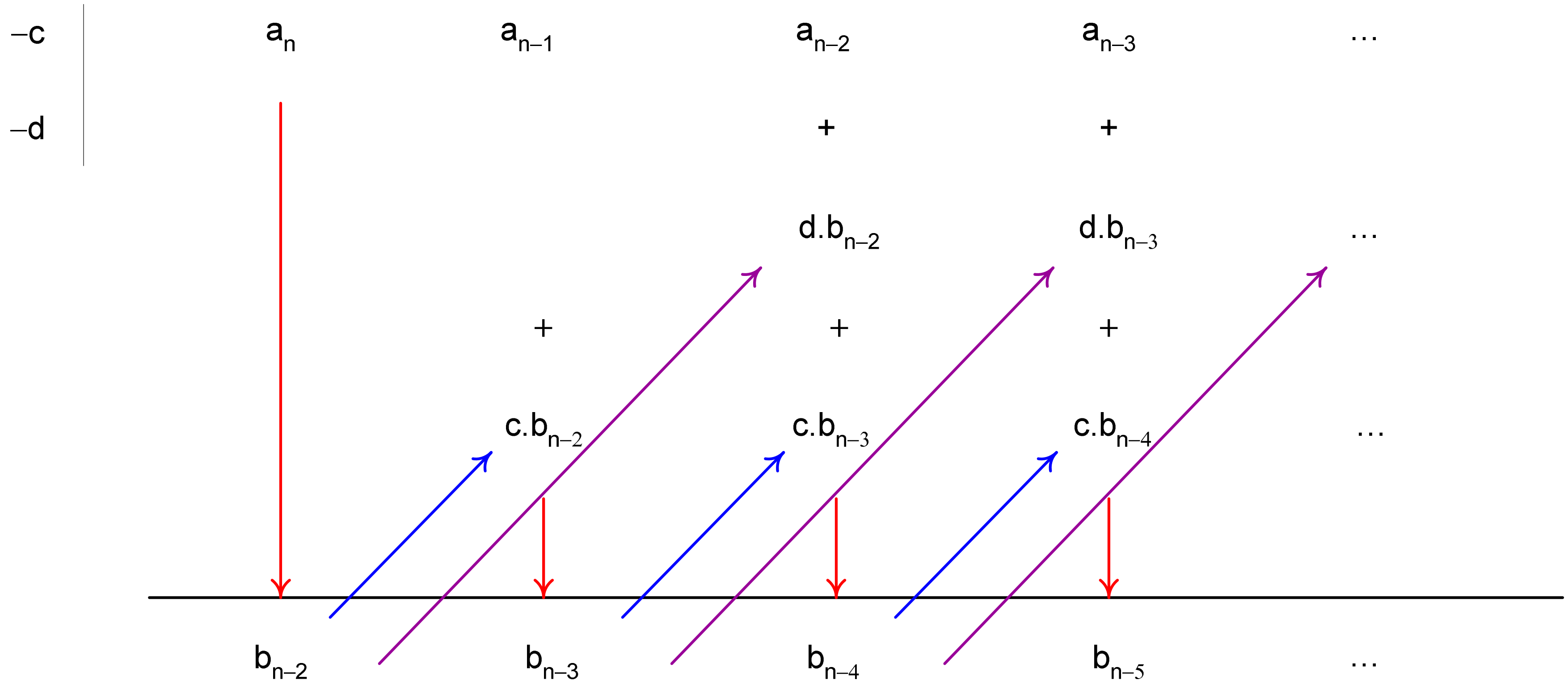
Hence $x^3 + 2x^2 - 3x - 1 = (x+4)(x^2 - 2x + 5) - 21$. Notice that the last two columns of the bottom row represents the remainder of the form $Ax + B$. In this case $A = 0$ which is as it should be (why? Hint: $f(x)$ is cubic, and we are diving by a quadratic. What does this imply?)

2) Divide $f(x) = 3x^3 + x^2 + 7x - 6$ by $x^2 + x + 3$

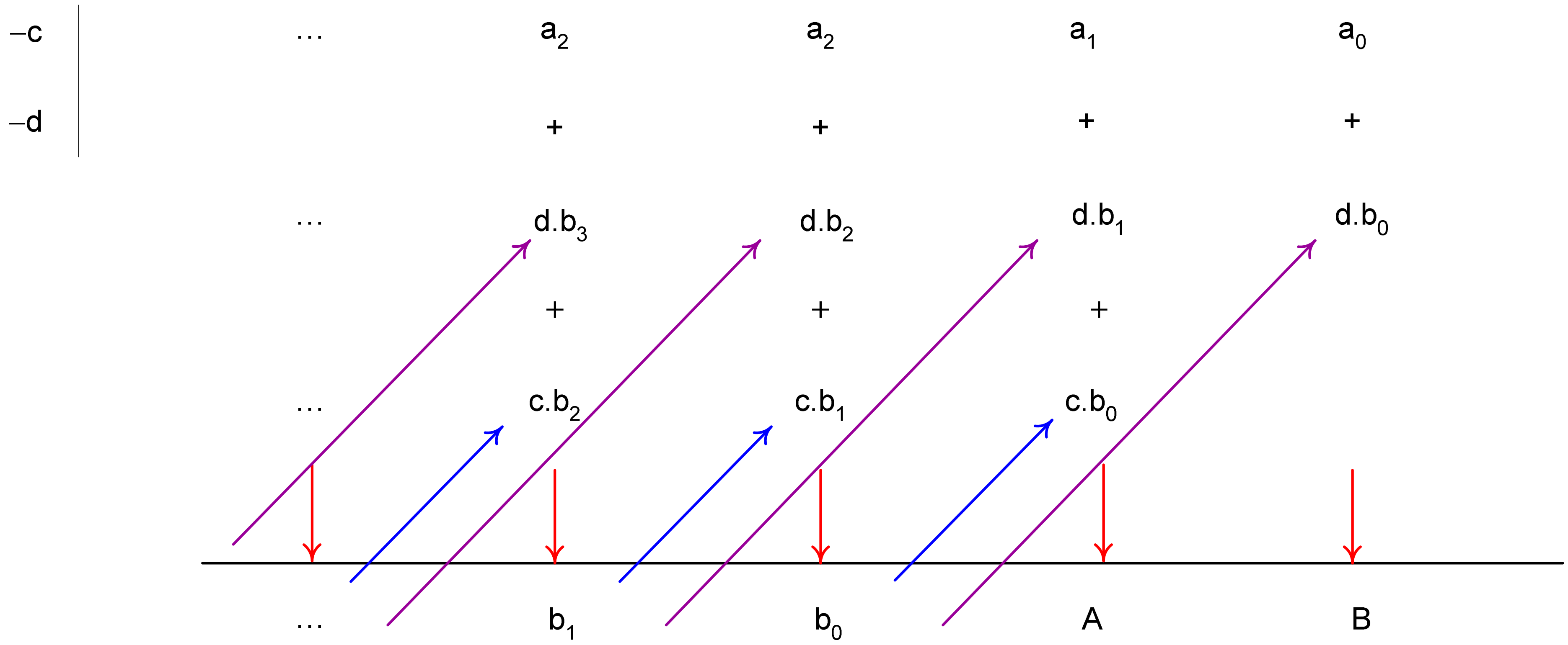
				$f(x)$		
			x^3	x^2	x	const
$g(x)$	$-x$	-1	3	1	7	-6
	$-$const	-3			-9	6
				-3	2	
			3	-2	0	0
		$Q(x)$:	x	const	A	B

Hence $3x^3 + x^2 + 7x - 6 = (3x - 2)(x^2 + x + 3)$, i.e. $x^2 + x + 3$ is a factor of $f(x)$.

The generalised table for this alternative synthetic division form is shown below for dividing $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ by $g(x) = x^2 + cx + d$, giving quotient $Q(x) = b_{n-2}x^{n-2} + b_{n-3}x^{n-3} + \cdots + b_1 x + b_0$ and remainder $Ax + B$.



Alternative version of the generalised synthetic division table for quadratic divisors (continued on next page)



Alternative version of the generalised synthetic division table for quadratic divisors

See *Short Division of Polynomials*, Li Zhou, *The College Mathematics Journal*, Vol. 40, No. 1 (Jan., 2009), pp. 44-46, for another compact form of synthetic division.

2.6.13 Quadratic divisors of the form $ax^2 + bx + c$

The following is adapted from "Using synthetic division by quadratics to find rational roots", M. W. Hutchinson, *The Mathematics Teacher*, vol. 64, no. 4 (April 1971), pp. 349-352.

In the previous section we performed synthetic division of a polynomial $f(x)$ by a quadratic divisor of the form $g(x) = x^2 + bx + c$. What if $g(x)$ is of the more general form $ax^2 + bx + c$? The usual process of long division will be the same but the synthetic division process is slightly different, and goes as follows:

- By the remainder theorem we have
$$f(x) = (ax^2 + bx + c)Q(x) + Ax + B.$$
- Factorise coefficient a to get
$$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)Q(x) + Ax + B.$$
- This is the same as
$$f(x) = \left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)Q^*(x) + Ax + B\ ,$$
where $Q^*(x) = aQ(x)$

What this means is that we can perform the standard synthetic division process using the divisor $g^*(x) = x^2 + \frac{b}{a}x + \frac{c}{a}$ to get $Q^*(x)$ as our quotient, and then recover $Q(x)$ by doing $Q^*(x)/a$.

<u>*Example*</u>

Consider using synthetic division to divide $f(x) = 2x^4 + 3x^2 - x + 1$ by $2x^2 - 4x + 10$. In this case we start by dividing $f(x)$ by $x^2 - 2x + 5$ to give us quotient $Q^*(x)$. Then we will divide $Q^*(x)$ by 2 to get the actual quotient $Q(x)$.

Using the alternative synthetic division process of the previous section we have

					$f(x)$		
			x^4	x^3	x^2	x	const
$g(x)$	$-x$	2	2	0	3	−1	1
	–const	–5			–10	–20	–5
				4	8	2	
			2	4	1	–19	–4
		$Q^*(x)$:	x^2	x	const	A	B

Hence the quotient $Q^*(x) = 2x^2 + 4x + 1$ and the remainder is $-19x - 4$. The actual quotient we want is $Q(x) = Q^*(x)/2$, i.e $Q(x) = x^2 + 2x + 1/2$. The factorisation is given by

$$2x^4 + 3x^2 - x + 1 = (2x^2 - 4x + 10\,)\left(x^2 + 2x + \frac{1}{2}\right) - 19x - 4\,.$$

2.6.14 Extending the idea of the remainder theorem: Divisors of degree m < n

We have seen the remainder theorem for linear and quadratic divisors to be

$$f(x) = (x - c)Q(x) + A$$

and

$$f(x) = (x - c)(x - d)Q(x) + Ax + B\,,$$

for constant $A, B, c, d \in \mathbb{R}$. The remainder theorem for cubic divisors is a logical extension of the above, i.e.

$$f(x) = (x - c)(x - d)(x - e)Q(x) + Ax^2 + Bx + C\,,$$

where $Ax^2 + Bx + C$ is the remainder when dividing $f(x)$ by $(x - c)(x - d)(x - e)$. In this case we would need to evaluate $f(x)$ at the three values $x = c, x = d, x = e$ in order to get three equation so as to solve for A, B, C.

For example, to find the the remainder when dividing $f(x) = x^4 - 2x^3 + x^2 + 3x - 6$ by $g(x) = x^3 - 7x + 6$ we first factorise $g(x)$ as $(x-2)(x-1)(x+3)$. This means that $f(x)$ can be expressed as

$$x^4 - 2x^3 + x^2 + 3x - 6 = (x-2)(x-1)(x+3)Q(x) + Ax^2 + Bx + C\ .$$

We now evaluate $f(-3), f(1)$, and $f(2)$ to find A, B, C:

$$f(-3) = 129 = 9A - 3B + C$$

$$f(1) = -3 = A + B + C$$

$$f(2) = 4 = 4A + 2B + C$$

which gives solutions

$$A = 8, B = -17, C = 6.$$

The remainder is therefore

$$R(x) = 8x^2 - 17x + 6.$$

The idea of division of a polynomial by linear, quadratic, and cubic divisors can be extended to divisors of any degree. So if $f(x)$ is a polynomial of degree n, and $g(x)$ is a polynomial divisor of degree m where $m < n$, then we can state a general remainder theorem as follows:

Let $f(x)$ be an n^{th} degree polynomial, and let $g(x) = (x - x_1)(x - x_2) \ldots (x - x_m)$ be an m^{th} degree polynomial such that $m < n$, and where $x_1, x_2, \ldots, x_m$ are real numbers. When $f(x)$ is divided by $g(x)$, the remainder is

$$R(x) = a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \cdots + a_1 x + a_0,$$

where $a_{m-1}, a_{m-2}, \ldots, a_1, a_0$ are constants, and are found by solving the systems of equations

$$\begin{array}{c} f(x_1) = a_{m-1}(x_1)^{m-1} + a_{m-2}(x_1)^{m-2} + \cdots + a_1 x_1 + a_0 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ f(x_m) = a_{m-1}(x_m)^{m-1} + a_{m-2}(x_m)^{m-2} + \cdots + a_1 x_m + a_0. \end{array}$$

In practice, matrices would be used to solve such a system.

All our work so far has focused on quadratic or linear divisors. However, the remainder theorem applies to divisors $g(x)$ of any degree provided that the degree of $g(x)$ is less than that of $f(x)$. In that case we have the most general representation of the structure of $f(x)$, a polynomial of degree n, upon division by $g(x)$, a polynomial of degree m, is

$$\frac{f(x)}{g(x)} = Q(x) + \frac{R(x)}{g(x)} \tag{13}$$

or

$$f(x) = g(x)Q(x) + R(x) \tag{14}$$

where $Q(x)$ is the quotient of degree $n - m$ and $R(x)$ is the remainder of degree $m - 1$ at most.

2.7 Selected studies on the remainder theorem

2.7.1 A study of the remainder theorem for linear divisors

This section is adapted from *Factor and Remainder Theorems: An Appreciation*, Michael Weiss, in *The Mathematics Teacher*, Vol. 110, No. 2 (September 2016), pp. 153-156.

By the remainder theorem we have $f(x) = (x - c)Q(x) + R$, for some divisor $(x - c)$ where c is a constant. When $x = c$ we have $f(c) = R$, and the above expresion becomes

$$f(x) = (x - c)Q(x) + f(c).$$

This expression is true for any value of x. For example, given that

$$x^3 - 7x^2 + 6x - 2 = (x - 2)(x^2 - 5x - 4) - 10$$

we know we obtain the value of –10 on the right hand side simply by evaluating the left hand side at $x = 2$. But we can also evaluate the expression above at $x = 0$ to get

$$\text{LHS:} \quad -2 \qquad , \qquad \text{RHS:} \quad (-2)(-4) - 10 = -2.$$

Let us therefore evaluate $f(x)$ at $x = a$. We get

$$f(a) = (a - c)Q(a) + f(c).$$

Doing some basic algebra this becomes

$$Q(a) = \frac{f(a) - f(c)}{a - c}.$$

The right hand side of this expression is the standard expression for the slope of the secant joining coordinates $\big(a, f(a)\big)$ and $\big(c, f(c)\big)$, and the value of this slope is given as the quotient $Q(a)$. Therefore, in general, given a linear divisor,

$$Q(x) = \frac{f(x) - f(c)}{x - c}$$

can be interpreted as the slope of the secant joining coordinates $\big(a, f(a)\big)$ and $\big(c, f(c)\big)$.

<u>*Example:*</u> We can use the expression for $Q(x)$ above to find the slope of the tangent line at any point x on the function $f(x)$. For example, let

$$f(x) = 2x^2 - 3x - 6.$$

Suppose we want to find the slope of the tangent at $x = 1$. Normally we would do this via calculus but in this case we can also find the slope by firstly dividing $f(x)$ by $(x - 1)$ to give

$$\frac{2x^2 - 3x - 6}{x - 1} = 2x - 1 - \frac{7}{x - 1}\,.$$

We now want to form

$$Q(x) = \frac{f(x) - f(c)}{x - c}\,,$$

so we bring the term $-7/(x - 1)$ over to the left hand side to obtain

$$Q(x) = \frac{2x^2 - 3x - 6 + 7}{x - 1} = 2x - 1. \qquad \{**\}$$

from which we obtain $Q(1) = 1$ which is the slope of the tangent line of $f(x)$ at $x = 1$.

Note that we cannot evaluate the rational function in {**} at $x = 1$ since we get ∞, but we can evaluate the right hand side of {**} at $x = 1$ to get an actual value. So {**} seems inconsistent. However, this is not actually the case. In the topic of calculus one learns about the operation of limits, what they are, how they work and when they work. It so happens that we can do this "limit" operation on the rational function in {**}, and when this is done the answer equals 1.

To see that this is indeed so we can evaluate the rational function in {**} at values which get closer and closer to 1:

x	0.99	0.999	0.9999	0.99999	1	1.00001	1.0001	1.001	1.01
$f(x)$	0.98	0.998	0.9998	0.99998	?	1.00002	1.0002	1.002	1.02

So we say that, in the limit as x approach 1, $(2x^2 - 3x - 6 + 7)/(x - 1)$ equals 1.

2.7.2 *A study of the remainder theorem for quadratic divisors*

This section is adapted from *Synthetic division for nonlinear factors*, K. W. Reimann, in *The Mathematics Teacher*, Vol 73, No. 3 (Mar 1980), pp. 231 – 233.

This example is to illustrate an interesting use of the remainder theorem. Consider wanting to evaluate $f(1)$ and $f(-3)$ for

$$f(x) = 18x^5 + 27x^4 - 32x^3 - 17x^2 + 20x - 4.$$

Evaluating this directly gives $f(1) = 12$ and $f(-3) = -1540$.

Consider now setting up a polynomial whose roots are –3 and 1, i.e. the values at which we want to evaluate $f(x)$. This polynomial will be our divisor. Therefore, let $g(x) = (x - 1)(x + 3) = x^2 + 2x - 3$. Now divide $f(x)$ by $g(x)$ in the usual way to get

$$f(x) = (x^2 + 2x - 3)(18x^3 - 9x^2 + 40x - 124) + 388x - 376.$$

Evaluating the remainder $R(x) = 388x - 376$ at $x = 1$ and $x = 3$ we obtain $R(1) = 12$ and $R(-3) = -1540$. In other words we get the same result via the remainder as by using $f(x)$ directly. This is not a coincidence.

Another example: Consider evaluating $f(-1), f(0)$ and $f(1)$ for

$$f(x) = x^6 - 5x + 2.$$

Evaluating this directly gives $f(-1) = 8, f(0) = 2$ and $f(1) = -2$.

Consider now the polynomial $g(x) = x(x - 1)(x + 1) = x^3 - x$ as the divisor to $f(x)$.

By division we get

$$f(x) = (x^3 - x)(x^3 + x) + x^2 - 5x + 2\,.$$

The remainder here is $R(x) = x^2 - 5x + 2$. Evaluating this at $x = -1, x = 0, x = 1$ gives $R(-1) = 8$, $R(0) = 2$ and $R(1) = -2$, again showing that we get the same result when evaluating by the remainder as we do when evaluating by $f(x)$.

To see why this works let us go back to the definition of the remainder theorem:

$$f(x) = g(x)Q(x) + R(x).$$

Now suppose that r_1 is a root of $g(x)$. Then $g(r_1) = 0$ and we have

$$f(r_1) = R(r_1).$$

This tells us that instead of evaluating $f(x)$ at $x = r_1$ we can evaluate $R(x)$ at $x = r_1$ provided we have set up our divisor $g(x)$ as a polynomial having at least one of its roots to be r_1.

2.7.3 A generalisation of the remainder theorem

The idea of this section is taken from *A generalisation of the remainder theorem and the factor theorem*, F. Laudano, in *International journal of mathematical education in science and technology*, published online 19 Sept. 2018 (DOI 10.1080/0020739X.2018.1522676)

We know that when a known polynomial $f(x)$ is divided by a linear divisor $(x - c)$ we have $f(x) = g(x)Q(x) + R$. The remainder theorem then says that $f(c) = R$, achieved by substituting $x = c$ into $f(x)$. But what if our divisor is $x^n - c$, where $n \in \mathbb{N}$? Can we substitute c for every instance of x^n and still get the valid remainder?

For example, dividing $x^8 - 7x^4 + x^3 - x^2 - 3x + 10$ by $x^3 + 1$ we get

$$x^8 - 7x^4 + x^3 - x^2 - 3x + 10 = (x^3 + 1)(x^5 - x^2 - 7x + 1) + 4x + 9.$$

Hence the remainder is $4x + 9$. If we were to use the remainder theorem as we know it we would have to factorise $g(x)$ into its one real root and two complex roots. Then we would substitute these roots into $f(x) = R(x)$ giving us three simultaneous equations to solve for three unknowns.

But suppose, instead, that we substitute $x^3 = -1$ into $f(x)$, i.e. we substitute every instance of x^3 by –1. Will this give us the same remainder? To see that it does, rewrite $f(x)$ in terms of x^3 where possible:

$$f(x) = (x^3)^2x^2 - 7(x^3)x + (x^3) - x^2 - 3x + 10.$$

Now substitute $x^3 = -1$ to get

$$f(-1|_{x^3}) = (-1)^2x^2 - 7(-1)x + (-1) - x^2 - 3x + 10\,,$$

where "$-1|_{x^3}$" means we are substituting –1 for every x^3 only. The above simplifies to

$$f(-1|_{x^3}) = 4x + 9$$

which is our remainder.

As another example consider dividing $f(x) = 8x^7 - 2x^6 + x^5 + 10x - 2$ by $g(x) = x^2 - 3x + 2$. Normally we would factorise $g(x)$ and then substitute $x = 1$ and $x = 2$ into $f(x) = R(x)$. However, let us instead substitute for x^2. In this case $x^2 = 3x - 2$. Then, in order to make such a substitution we need to rewrite $f(x)$ in terms of x^2 where possible:

$$f(x) = 8(x^2)^3x - 2(x^2)^3 + (x^2)^2x + 10x - 2,$$

from which we obtain

$$f_1(x) = 8(3x - 2)^3x - 2(3x - 2)^3 + (3x - 2)^2x + 10x - 2,$$

which simplifies to

$$f_1(x) = 216x^4 - 477x^3 + 384x^2 - 122x + 14.$$

Since our divisor is quadratic we know that the remainder has to be at most linear, but $f_1(x)$ is quartic, hence $f_1(x)$ is not our remainder. So we continue substituting $x^2 = 3x - 2$ until $f(x)$ is reduced to at most a linear expression. Therefore

$$f_2(x) = 216(3x - 2)^2 - 477(3x - 2)x + 384(3x - 2) - 122x + 14$$

which simplifies to

$$f_2(x) = 513x^2 - 608x + 110.$$

And again (why?)

$$f_3(x) = 513(3x - 2) - 608x + 110,$$

which simplifies to be the remainder

$$f_3(x) = 931x - 916.$$

This is indeed the result we would get for the remainder if we used long division.

What this shows is that given a polynomial $f(x)$ of degree n:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

and a polynomial divisor $g(x)$ of degree $m < n$:

$$g(x) = b_m x^m + b_{m-1}x^{m-1} + b_{m-2}x^{m-2} + \cdots + b_1 x + b_0$$

we can find the remainder when dividing $f(x)$ by $g(x)$ simply by rewriting $f(x)$ in terms of powers of x^m and then substituting for x^m from $g(x)$, i.e substituting

$$x^m = -\frac{b_{m-1}}{b_m}x^{m-1} - \frac{b_{m-2}}{b_m}x^{m-2} - \cdots - \frac{b_1}{b_m}x - \frac{b_0}{b_m}.$$

The proof that this works in general involves some advanced maths which is why I have left this out. Refer to the original paper if you are interested in this. A relatively straighforward proof for the case when $g(x) = x^m - c$ can be found in *An Extension of the Remainder Theorem*, D. C. Armstrong, *The Mathematical Gazette*, Vol. 55, No. 394 (Dec., 1971), pp. 419-420.

The advantage of the analysis above is that we don't have to factorise the divisor $g(x)$. This is not only useful but of crucial help when $g(x)$ does not have real roots. For example, if $g(x) = x^3 + 1$, then $g(x)$ has one real root and two complex roots, so how would we apply the remainder using the complex roots?

Example 1: To divide $f(x) = 7x^5 + 2x^3 + 3x^2 - 5x + 12$ by $-x^3 + 2$ we can replace every instance of x^3 in $f(x)$ by 2. Hence

$$f(x) = 7x^3x^2 + 2x^3 + 3x^2 - 5x + 12.$$

Therefore

$$f(2|_{x^3}) = 7(2)\,x^2 + 2(2) + 3x^2 - 5x + 12 = 17x^2 - 5x + 16.$$

Since the degree of $f(2|_{x^3})$ is less than that of the divisor, $f(2|_{x^3})$ is the remainder we want. Hence

$$R(x) = 17x^2 - 5x + 16.$$

Example 2: To divide $f(x) = 8x^5 + 2x^4 + 6x^2 - 5x + 1$ by $2x^3 + x - 2$ we can replace every instance of x^3 in $f(x)$ by $(2 - x)/2$. Hence $f(x) = 8x^3x^2 + 2x^3x + 6x^2 - 5x + 1$. Therefore

$$f\left(\frac{2-x}{2}\bigg|_{x^3}\right) = f_1(x) = 8\left(\frac{2-x}{2}\right)x^2 + 2\left(\frac{2-x}{2}\right)x + 6x^2 - 5x + 1$$

$$= -4x^3 + 13x^2 - 3x + 1$$

Continuing

$$f_1\left(\frac{2-x}{2}\bigg|_{x^3}\right) = f_2(x) = -4\left(\frac{2-x}{2}\right) + 13x^2 - 3x + 1$$

$$= 13x^2 - x - 3.$$

Since the degree of $f_2(x)$ is less than that of the divisor, $f_2(x)$ is the remainder we want. Hence

$$R(x) = 13x^2 - x - 3.$$

Example 3: To divide $f(x) = 2x^5 - x^3 + 6x^2 - x + 3$ by $x^2 - x + 3$ we can replace every instance of x^2 in $f(x)$ by $x - 3$. Hence

$$f(x) = 2(x^2)^2x - x^2x + 6x^2 - x + 3.$$

Therefore

$$f([x-3]|_{x^2}) = f_1(x) = 2(x-3)^2x - (x-3)x + 6(x-3) - x + 3\,,$$

$$= 2x^3 - 13x^2 + 26x - 15.$$

And again

$$f_1([x-3]|_{x^2}) = f_2(x) = 2x(x-3) - 13(x-3) + 26x - 15\,,$$

$$= 2x^2 + 7x + 24.$$

And finally

$$f_2([x-3]|_{x^2}) = f_3(x) = 2(x-3) + 7x + 24\,,$$

$$= 9x + 18.$$

Hence our remainder is $R(x) = 9x + 18$.

2.7.4 *Divisibility rules*

This section is taken, and extended, from "An application of the remainder theorem", R. W. Wagner, *The American Mathematical Monthly*, 54(2) (Feb 1947), p106.

We have seen the general factorisation of $f(x)$ to be $f(x) = (x - c)Q(x) + f(c)$. This can be recast as

$$\frac{f(x)}{x-c} = Q(x) + \frac{f(c)}{x-c}, \qquad [*]$$

where $x \neq c$ (but we can have $x \to c$). Now, let $f(x)$ by an n^{th} degree polynomial whose coefficients are any one of the positive integers 0 to 9. For example, $f(x) = 2x^5 + 3x^4 + 6x^2 + 1$. Then $f(10)$ is the ordinary decimal representation of a number, i.e.

$$f(10) = 2 \times 10^5 + 3 \times 10^4 + 6 \times 10^2 + 1 = 230601.$$

Also, notice that $f(1)$ is the sum of the coefficients of $f(x)$, i.e.

$$f(1) = 2 \times 1^5 + 3 \times 1^4 + 6 \times 1^2 + 1 = 12.$$

Notice that 12 is simply the sum of the digits of the number 230601.

Now, setting $c = 1$ we can write [*]as

$$\frac{f(10)}{9} = Q(10) + \frac{f(1)}{9}. \qquad [[*]]$$

The quotient $Q(10)$ is an integer, therefore if $f(1)$ is divisible by 9, $f(1)/9$ will be an integer, implying that $f(10)/9$ will be integer, and therefore $f(10)$ is also divisible by 9.

This brings us to a very simple criteria/test for deciding whether or not a number is divisible by 9:

If the sum of the digits of a number is divisible by 9,
the number itself is divisible by 9.

For example, 540 is divisible by 9 since $5 + 4 = 9$ which is divisible by 9.

On the other hand, 219 is not divisible by 9 since $2+1+9=12$ which is not divisible by 9. But 7632 is divisible by 9 since $7+6+3+2=18$ which is divisible by 9. However, 32118 is not divisible by 9 since $3+2+1+1+8=15$ is not divisible by 9.

We can also develop a divisibility rule for dividing by 3. If we multiply [[*]] by 3 we have

$$\frac{f(10)}{3} = 3Q(10) + \frac{f(1)}{3}.$$

Here we have $3Q(10)$ to be integer, therefore if $f(1)$ is divisible by 3, $f(1)/3$ will be an integer. This implies that $f(10)/3$ will be integer, and therefore $f(10)$ is also divisible by 3. Hence

if the sum of the digits of a number is divisible by 3,
the number itself is divisible by 3.

For example, 312 is divisible by 3 since $3+1+2=6$ which is divisible by 3. Similarly 123 and 231 are divisible by 3. On the other hand, 611 is not divisible by 3 since $6+1+1=8$ which is not divisible by 3. Neither are 1112, 10381, or 5383118. But 4830 is divisible by 3 since $44+8+3+0=15$ which is divisible by 3. Similarly, any permutation of 4830 will be divisible by 3, viz 8403, 4380,4308, 3038, etc.

We can also develop a divisibility rule for dividing numbers by 11. To do this let $x=10$ and $c=-1$ in [*]. Then we have

$$\frac{f(10)}{11} = Q(10) + \frac{f(-1)}{11}.$$

Now, $f(-1)$ is the alternating sum of the coefficients, i.e. $f(-1) = \pm a_n \mp a_{n-1} \pm \cdots \pm a_1 \mp a_0$. Again, $Q(10)$ is an integer, so

if the alternating sum of the digits of a number is divisible by 11,
the number itself is divisible by 11.

Here we have to be careful which coefficients are positive and which are negative. Then we have

- 2728 is divisible by 11 since $-2+7-2+8=11$ is divisible by 11 (note that it doesn't matter if we do $2-7+2-8=-11$);
- 3141 is not divisible by 11 since $-3+1-4+1=-5$ is not divisible by 11;

- 31416 is divisible by 11 since $3 - 1 + 4 - 1 + 6 = 11$ is divisible by 11;
- 296436 is not divisible by 11 (confirm this).

Notice that all of our arithmetic above has been based on decimal (base 10) numbers, and $f(10)$ represented a number in decimal (base 10) form. Let x of [*] now represent the base number system we are working in. Then $f(7)$ means we are working in base 7. We can develop divisibility rules for numbers in based 7 as well as other bases.

Theerefore, letting $c = 1$ in [*] we have

$$\frac{f(7)}{6} = Q(7) + \frac{f(1)}{6},$$

where $f(7)$ represents the sum of the coefficients, base 7, of $f(x)$, i.e. $f(7) = a_n 7^n + a_{n-1}7^{n-1} + \cdots + a_1 7 + a_0$. Since $Q(7)$ base 7 is an integer, if $f(1)/6$ base 7 is an integer then $f(1)$ is divisible by 6.

Example 1: Let 16320 be an integer base 7. Then, adding the digits up normally (i.e. in base 10) we have $1 + 6 + 3 + 2 + 0 = 12$ base 10, which is 15 base 7. But 15 base 7 is divisible by 6 (i.e. $15/6$ base $7 = 2$). Hence 16320 base 7 is divisible by 6.

Example 2: The number 8395 is not a base 7 number (what base is it?). So we first convert it into a base 7 number, which gives 33322. The sum of the digits of this last number is 13 base 10 which is 16 base 7. Since 16 is not divisible by 6 in the base 7 number system, 33322 base 7 is not divisible by 6.

So we have that

if the sum, base 7, of the digits of a number, base 7, is divisible by 6,
the number itself is divisible by 6.

Exercise: Develop divisibility rules for numbers base 3, 5, and 8, when dividing by 2, 4, and 7 respectively, as well as when dividing by 11 base 3, 11 base 5, and 11 base 8 respectively.

2.8 Aspects of factorisation IV: The factor theorem

Here we study something called the factor theorem which is a consequence of the remainder theorem under a particular condition.

2.8.1 The factor theorem

Consider $10 = 2 \times 5$. The numbers 2 and 5 are said to be factors of 10. A factor is a divisor which goes completely into (i.e. without remainder) the number being divided.

Returning to $f(x) = (x - c)Q(x) + R$, we see that if $R = 0$ then

$$f(x) = (x - c)Q(x). \tag{15}$$

where $Q(x)$ is of degree one less than $f(x)$. In this case $(x - c)$ is said to be a factor of $f(x)$. This means that $(x - c)$ completely divides $f(x)$. The factor theorem can therefore be stated as follows:

Factor theorem (for linear divisors)

Let $f(x)$ be an n^{th} degree polynomial and let us divide $f(x)$ by $(x - c)$, where c is a real number. If $f(c) = 0$ then $(x - c)$ is a factor of $f(x)$.

The factor theorem allows us to determine whether or not a linear divisor is a factor of a polynomial simply by substituting some value for x into that polynomial.

As an example, consider $f(x) = 6x^4 - 42x^3 + 102x^2 - 102x + 36$. We can use the factor theorem to tell whether of not the following are factors of $f(x)$: $(x - 1)$, $(x + 1)$ and $(x - 2)$:

- If $x - 1$ is a factor of $f(x)$ then setting $x = 1$ implies $f(1) = 0$:

  $$f(1) = 6 - 42 + 102 - 102 + 36 = 0,$$

  hence $x - 1$ is indeed a factor of $f(x)$. Actually, $(x - 1)^2$ is a factor of $f(x)$ implying that $x - 1$ is a repeated factor. We will study how to find repeated factors in section 2.8.3;

- If $x + 1$ is a factor of $f(x)$ then setting $x = -1$ implies $f(-1) = 0$:

  $$f(-1) = 6 + 42 + 102 + 102 + 36 \neq 0,$$

  hence $x + 1$ is not a factor of $f(x)$;

- If $x - 2$ is a factor of $f(x)$ then setting $x = 2$ implies $f(2) = 0$:

$$f(2) = 96 - 336 + 408 - 204 + 36 = 0,$$

hence $x - 2$ is indeed a factor of $f(x)$.

The proof of the factor theorem is quite straightforward, and relies on analysing the remainder theorem from two directions, namely i) what $(x - c)$ represents when $R = 0$, and ii) what the value of R is if $(x - c)$ is already a factor of $f(x)$.

Proof: Let $f(x) = (x - c)Q(x) + R$. If $R = 0$ then

$$f(x) = (x - c)Q(x)$$

automatically implying that $(x - c)$ is a factor of $f(x)$. On the other hand, if $(x - c)$ is a factor of $f(x)$ this implies that $R = 0$ simply by definition of what a factor is. ■

Let us now suppose that in (15) the quotient $Q(x)$ has a factor $(x - d)$. We then have

$$f(x) = (x - c)(x - d)P(x) \tag{16}$$

(note that if $f(x)$ is of degree n, $P(x)$ is of degree $n - 2$). We then have the following:

- when $x = c$ the factor theorem tells us that $f(c) = 0$,

and

- when $x = d$ the factor theorem tells us that $f(d) = 0$.

As an example consider

$$f(x) = 2x^4 - 5x^3 + 4x^2 - 5x + 2.$$

One value of x which makes $f(x) = 0$ is $x = 2$, hence $(x - 2)$ is a factor of $f(x)$. Another value of x which makes $f(x) = 0$ is $x = 1/2$. So $(2x - 1)$ is also a factor of $f(x)$. Hence $f(x)$ can be written as

$$f(x) = (x - 2)(2x - 1)P(x).$$

The factor theorem for quadratic divisors can therefore be stated as follows:

Factor theorem (for quadratic divisors)

Let $f(x)$ be an n^{th} degree polynomial and let us divide $f(x)$ by $(x-c)(x-d)$, where $c, d \in \mathbb{R}$. If $f(c) = 0$ and $f(d) = 0$ then $(x-c)(x-d)$ is a quadratic factor of $f(x)$.

We do not have to stop at equation (16) when it comes to finding factors of $f(x)$. We can continue by trying to find factors of $P(x)$ (if they exists). Hence, if $P(x)$ contains a factor $(x-e)$ we have

$$f(x) = (x-c)(x-d)(x-e)S(x)$$

where $S(x)$ is of degree $n-3$. We might then find that $S(x)$ has another factors of $(x-c)$ in which case we now have

$$f(x) = (x-c)^2(x-d)(x-e)T(x)$$

where $T(x)$ is of degree $n-4$. How far can we go with this factorization? Can all polynomials be factorized into multiples of linear factors? The answer is yes, but only if we are working in the domain of complex numbers (in other words, only if our factors are allowed to be complex when necessary). If we work only in the domain of real numbers the answer is no. The best we can do is to factorise all linear factors from $f(x)$ except for quadratic equations which cannot be factorised due to $\Delta < 0$.

An example of this last situation is that of the factorisation of

$$f(x) = x^6 - 4x^5 + 6x^4 - 8x^3 + 13x^2 - 12x + 4.$$

The aim of this example is not to go through how to factorise$f(x)$ but simply to see how far it can be factorised. The complete factorization to $f(x)$ is

$$f(x) = (x-2)(x-1)^3(x^2+x+2).$$

Here we cannot factorise $f(x)$ into only linear factors because the discriminant Δ of the quadratic is negative. Similarly, the complete factorization of the polynomial

$$f(x) = 8x^8 + 12x^7 + 30x^6 - 27x^5 - 102x^4 - 225x^3 - 242x^2 - 111x - 18$$

is

$$f(x) = (x-2)(2x+1)^3(x^2+x+3)^2,$$

and again it is the case that the discriminant of the quadratic is negative, implying that we cannot reduce $f(x)$ any further.

So in general the most factorized version (in the domain of real numbers) of a general n^{th} degree polynomial $f(x)$ will be

$$f(x) = (x - x_1)^{m_1}(x - x_2)^{m_2} \dots (x - x_k)^{m_k}$$
$$\times (a_1x^2 + b_1x + c_1)^{p_1}(a_2x^2 + b_2x + c_2)^{p_2} \dots (a_kx^2 + b_kx + c_k)^{p_k}$$

This is the *fundamental theorem of algebra*. It says that any polynomial whose coefficients are real can be factorized in the real domain into (possibly repeated) linear factors and (possibly repeated) irreducible quadratics.

Example 1: To find out if $x - 1$ is a factor of $f(x) = x^3 - 7x + 6$ we simply substitute $x = 1$ into $f(x)$, whence

$$f(-1) = -1 + 7 + 6 \neq 0.$$

Hence $x - 1$ is not a factor of $f(x)$.

Example 2: To find out if $2x - 1$ is a factor of $f(x) = 2x^4 - x^3 - 6x^2 + 5x - 1$ we substitute $x = 1/2$ into $f(x)$ to obtain

$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{16}\right) - \frac{1}{8} - 6\left(\frac{1}{4}\right) + 5\left(\frac{1}{2}\right) - 1 = 0.$$

Hence $2x - 1$ is a factor of $f(x)$.

Example 3: If $x - 2$ is a factor of $f(x) = ax^2 - 12x + 4$ we can find a as follows:

$$f(2) = 4a - 12(4) + 4 = 4a - 44 = 0.$$

Hence $a = 11$.

Example 4: Consider factorising $f(x) = 27x^3 - 1$. If this can be factorised there must be a value of x such that $f(x) = 0$. Solving $f(x) = 0$ directly gives $x = 1/3$. So we have

$$f(x) = (3x - 1)Q(x),$$

where $Q(x)$ is a quadratic. We can find $Q(x)$ as usual, by long division or by expanding and comparing coefficients (left as an exercise), so that we obtain

$$f(x) = (3x - 1)(9x^2 + 3x + 1).$$

The discriminant of the quadratic is $\Delta = 3^2 - 4(9)(1) = -27$, so the quadratic cannot be factorised. Hence the complete factorisation for $f(x)$ in the real domain is $f(x) = (3x - 1)(9x^2 + 3x + 1)$.

Exercise: Factorise, as far as possible, $f(x) = x^3 - a^3$ where $a \in \mathbb{R}$.

Example 5: This example uses the imaginary number i. If you have not yet met this you can come back to this example at a later time.

To show that $g(x) = x^2 + 3$ is a factor of $f(x) = x^3 - x^2 + 3x - 3$ we first factorise $g(x)$. The values of x for which $g(x) = 0$ are found by solving $x^2 + 3 = 0$. Hence $x = \pm i\sqrt{3}$, and $g(x) = (x - i\sqrt{3})(x + i\sqrt{3})$.

We now substitute $x = \pm i\sqrt{3}$ into $f(x)$ to obtain

$$f(i\sqrt{3}) = (i\sqrt{3})^3 - (i\sqrt{3})^2 + 3(i\sqrt{3}) - 3 = 0,$$

and

$$f(-i\sqrt{3}) = (-i\sqrt{3})^3 - (-i\sqrt{3})^2 + 3(-i\sqrt{3}) - 3 = 0,$$

Hence $x^2 + 3$ is a factor of $f(x)$, and the complete factorisation for $f(x)$ is

$$f(x) = (x^2 + 3)(x - 1),$$

which can be shown by the usual methods (left as an exercise).

Exercise: Find the constant m for which $x^2 + 1$ is a factor of $f(x) = mx^4 + x^2 - 1$

Example 6: Find a constant p such that $x^2 + 2$ is a factor of $f(x) = x^4 - 6x^2 + p$. Hence factorise $f(x)$.

Solution: If $x^2 + 2$ is a factor of $f(x)$ then by the factor theorem

$$x^4 - 6x^2 + p = (x^2 + 2)Q(x) = (x^2 + 2)(x^2 + bx + c).$$

All we now have to do is to expand the right hand side and compare coefficients. The expansion is left as an exercise.

Comparing coefficients gives

$$x^3: \qquad b = 0 \qquad \text{(a)}$$

$$x^2: \qquad -6 = 2 + c \qquad \text{(b)}$$

$$x: \qquad 0 = 2b \qquad \text{(c)}$$

$$\text{constants}: \qquad p = 2c \qquad \text{(d)}$$

From (b) we have $c = -8$ and from (d) we have $p = 2(-8) = -16$. A useful thing to notice, which acts as a check on our work, is that (a) and (c) agree. Hence

$$f(x) = x^4 - 6x^2 - 16 = (x^2 + 2)(x^2 - 8)$$

(note that example 5 could have been solved this way).

Example 7: Show that the quadratic factor of $f(x) = 2x^4 + x^3 - x^2 + 8x - 4$ has no real roots.

Solution: Since $f(x) = 2x^4 + x^3 - x^2 + 8x - 4$ has a quadratic factor with no real roots we know that $f(x)$ can be factorised either as one quadratic and two linear factors, or as two quadratic factor (both irreducible), i.e.

$$2x^4 + x^3 - x^2 + 8x - 4 = (x - a)(x - b)(cx^2 + dx + e)\,, \qquad (*)$$

or

$$2x^4 + x^3 - x^2 + 8x - 4 = (ax^2 + bx + c)(dx^2 + ex + f)\,. \qquad (**)$$

Let us see if we can find any linear factors of $f(x)$. Testing this for a few value of x we have $f(1) = 6$, $f(-1) = 12$, $f(2) = 48$, $f(-2) = 0$. Hence $x + 2$ is a factor of $f(x)$.

By our knowledge of the properties of quadratic we know there must be another linear factor which is real (the other factor cannot be complex. Why?). So, continuing to test we find that $f(1/2) = 0$ implying that $2x - 1$ is also a factor of $f(x)$. Therefore

$$2x^4 + x^3 - x^2 + 8x - 4 = (x + 2)(2x - 1)(dx^2 + ex + f)\,.$$

By expanding the RHS and comparing coefficients (left as an exercise) we obtain $2 = 2d$, $1 = 2e + 3d$, $-1 = 2f + 3e - 2d$, from which we have $d = 1$, $e = -1$, and $f = 2$. Hence

$$2x^4 + x^3 - x^2 + 8x - 4 = (x + 2)(2x - 1)(x^2 - x + 2).$$

The discriminant of the quadratic is $\Delta = -7$, implying there are no real roots to the quadratic factor.

Example 8: Two cubic polynomials are defined by

$$f(x) = x^3 + (a-3)x + 2b \text{ and } g(x) = 3x^3 + x^2 + 5ax + 4b\ ,$$

where a and b are constants. Given that $f(x)$ and $g(x)$ have a common factor of $x-2$, show that $a = -4$ and find the value of b.

Solution:

If the common factor to $f(x)$ and $g(x)$ is $(x-2)$ then we can write

$$f(x) = x^3 + (a-3)x + 2b = (x-2)Q(x)$$

and

$$g(x) = 3x^3 + x^2 + 5ax + 4b = (x-2)P(x)\ .$$

Letting $x = 2$ we obtain

$$f(2) = 8 + (a-3)x + 2b = 0$$

and

$$g(2) = 24 + 4 + 10a + 4b = 0\ .$$

Solving these two equations we obtain $a = -4$ and $b = 3$. Alternatively we can solve as follows: By the factor theorem we have

$$f(x) = x^3 + (a-3)x + 2b = (x-2)(rx^2 + sx + t)$$

and

$$g(x) = 3x^3 + x^2 + 5ax + 4b = (x-2)(lx^2 + mx + n)$$

where l, m, n, r, s, t are constants. We now expand the right hand side of both of these equations and compare coefficients. Therefore

$$f(x) = x^3 + (a-3)x + 2b = rx^3 + sx^2 - 2rx^2 + tx - 2sx - 2t$$

and

$$g(x) = 3x^3 + x^2 + 5ax + 4b = lx^3 + mx^2 - 2lx^2 + nx - 2mx - 2n.$$

Comparing coefficients we obtain, for $f(x)$

x^3:	$r = 1$			
x^2:	$0 = s - 2r$	$\Rightarrow$	$s = 2$	
x:	$a - 3 = t - 2s$	$\Rightarrow$	$a = t - 1$	(i)
Constant:	$2b = -2t$	$\Rightarrow$	$b = -t$	(ii)

and for $g(x)$ we obtain

x^3:	$l = 3$			
x^2:	$1 = m - 2l$	$\Rightarrow$	$m = 7$	
x:	$5a = n - 2m$	$\Rightarrow$	$5a + 14 = n$	(iii)
Constant:	$4b = -2n$	$\Rightarrow$	$-2b = n$	(iv)

Combining (i) and (ii), and (iii) and (iv), give $a = -b - 1$ and $5a + 14 = -2b$ which we can solve as usual to obtain $a = -4$ and $b = 3$.

More difficult examples

Here we will go through some more involved examples. Don't necessarily expect to know how to solve these first time around. Rather, spend some time studying these examples in order to learn how it is that the remainder theorem is being applied as part of the process of solving these problem.

1) This question is adapted from question 4, STEP 1, 2007: Show that $x + b + c$ is a factor of $f(x) = x^3 - 3xbc + b^3 + c^3$. Hence, factorise $f(x)$ in the form $(x + b + c)Q(x)$, where $Q(x)$ is a quadratic expression. Show that $2Q(x)$ can be written as the sum of three expressions, each of which is a perfect square.

It is given that the equation $ay^2 + by + c = 0$ and $by^2 + cy + a = 0$ have a common root k, where $a, b \in \mathbb{R}$, $a, b \neq 0$ and $ac \neq b^2$. Show that

$$(ac - b^2)k = bc - a^2$$

and determine a similar expression for k^2. Hence show that

$$(ac - b^2)(ab - c^2) = (bc - a^2)^2\ ,$$

and that

$$(a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc) = 0\ .$$

Solution: If $x + b + c$ is a factor of $f(x)$ the we substitute $x = -b - c$ into $f(x)$ to confirm that $f(-b - c) = 0$. Hence

$$\begin{aligned} f(-b-c) &= (-b-c)^3 - 3(-b-c)bc + b^3 + c^3\ , \\ &= -(b^3 + 3b^2c + 3bc^2 + c^3) + 3b^2c + 3bc^2 + b^3 + c^3\ , \\ &= 0\ . \end{aligned}$$

So $(x + b + c)$ is a factor of $f(x)$. By the factor theorem we have

$$f(x) = (x + b + c)Q(x) = (x + b + c)(x^2 + px + q)\ ,$$

where we need to find p and q. Hence

$$x^3 - 3xbc + b^3 + c^3 = (x + b + c)(x^2 + px + q)\ .$$

We can find p and q by long division or by expanding and comparing coefficients. Performing the latter gives, after expansion (left as an exercise),

$$\begin{aligned} x^2: &\qquad 0 = p + b + c &\qquad (1) \\ x: &\qquad -3bc = q + p(b + c) &\qquad (2) \\ \text{constants}: &\qquad b^3 + c^3 = q(b + c) &\qquad (3) \end{aligned}$$

From (1) we have $p = -b - c$. We can then substitute this into (2) to find q, or we can find q from (3) by long division. Either way, confirm that $q = b^2 - bc + c^2$ (notice that you can check this answer for q by substituting it back into (2) or (3)).

Hence

$$f(x) = (x + b + c)(x^2 - xb - xc + b^2 - bc + c^2).$$

[note that we could think to group in term of $(b+c)$:

$x^2 - 2x(b+c) + b^2 + 2bc + c^2 + x^2 + b^2 + c^2 - 4bc$

$$= [x-(b+c)]^2 + x^2 + b^2 - 2bc + c^2 - 2bc\ ,$$

$$= [x-(b+c)]^2 + x^2 + (b-c)^2 - 2bc\ ,$$

but this is not in the form of three perfect squares].

Now we form $2Q(x)$ and do some algebra to show that it is composed of three perfect squares.

$$\begin{aligned} 2Q(x) &= 2(x^2 - xb - xc + b^2 - bc + c^2)\ , \\ &= 2x^2 - 2xb - 2xc + 2b^2 - 2bc + 2c^2\ . \end{aligned}$$

What we now need to do is to look to group terms in such a way as to give three perfect squares, and this grouping is shown below:

$$\begin{aligned} 2Q(x) &= x^2 - 2xb + b^2 + x^2 - 2xc + c^2 + b^2 - 2bc + c^2\ , \\ &= (x-b)^2 + (x-c)^2 + (b-c)^2. \end{aligned}$$

For the next part of the question, if $ay^2 + by + c = 0$ and $by^2 + cy + a = 0$ have a common root k then we have

$$ak^2 + bk + c = 0 \qquad (4)$$

and

$$bk^2 + ck + a = 0. \qquad (5)$$

We are looking for two answers, one in terms of k and one in terms of k^2. The easiest thing to do is to eliminate k^2 and k respectively from these equation. Hence

$$\begin{array}{lcrcl} b \times (4): & & abk^2 + b^2k + bc & = & 0 \\ & - & & & \\ a \times (5): & & bak^2 + cak + a^2 & = & 0 \\ \hline & & (b^2 - ac)k + bc - a^2 & = & 0 \end{array}$$

Hence

$$(ac - b^2)k = bc - a^2\ . \qquad (6)$$

Similarly $c \times (4) - b \times (5)$ gives

$$(ac - b^2)k^2 = ab - c^2\ . \qquad (7)$$

To get the final expression required we need to eliminate k and k^2 from (6) and (7). We can do this by squaring (6) and equating it to (7):

$$\left(\frac{bc - a^2}{ac - b^2}\right)^2 = \frac{ab - c^2}{ac - b^2},$$

implying

$$(bc - a^2)^2 = (ab - c^2)(ac - b^2).$$

Multiplying all this out, and simplifying, gives

$$a^3 - 3abc + b^3 + c^3 = 0.$$

Comparing this equation with the original equation at the start of the question we see that we can put $x = a$ into the original equation, and therefore into its factorisation. Hence we obtain

$$(a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc) = 0.$$

2.8.2 A study of the factor theorem

Let us consider how to create a fully factorisable polynomial from any starting polynomial. Suppose we want to invent a 7th degree polynomial

$$f(x) = a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

without simply writing down a product of seven linear factors. It is highly unlikely we will be able to invent the relevant (non-trivial) coefficients which make $f(x)$ fully factorisable. However, we can use the equation $f(x) = (x - c)Q(x) + R$ to help us create such a polynomial simply by writing

$$f(x) - R = (x - c)Q(x)$$

and repeating this as many times as needed. The left hand side is a polynomial, say $p(x)$, of the same degree as $f(x)$, and the right hand side says that we can factor out $(x - c)$ from $p(x)$. So we can create a polynomial $p(x)$ with at least one factor of our choice.

For example, suppose $f(x) = 27x^7 - 54x^6 + 117x^5 - 80x^4 - 75x^3 + 34x^2 + 27x + 4$ and we want to create a polynomial which has factor $(x + 1)$. We first divide $f(x)$ by $(x + 1)$ to obtain

$$f(x) = (x + 1)Q(x) - 192,$$

and then create a polynomial $p_1(x)$ with factor $(x+1)$ to be

$$p_1(x) = f(x) + 192 = (x+1)Q_1(x).$$

We can find $Q_1(x)$ as usual (by comparing coefficients or by long division) to be

$$Q_1(x) = 27x^6 - 81x^5 + 198x^4 - 278x^3 + 203x^2 - 169x + 196.$$

Suppose we now want to create a polynomial from $Q_1(x)$ which has factor $(x-3)$. Again we divide $Q_1(x)$ by $(x-3)$ to obtain

$$f(x) + 192 = (x+1)[(x-3)Q_2(x) + 10048].$$

Thus we can create a polynomial $p_2(x)$ with factors $(x+1)(x-3)$ to be

$$p_2(x) = f(x) + 192 - 10048(x+1) = (x+1)(x-3)Q_2(x),$$

where we can again find $Q_2(x)$ as usual. Carrying on in this way, dividing every subsequent quotient by any divisor we choose, we can convert our divisor into a factor of a polynomial $p_n(x)$, and thus create a completely factorisable polynomial of degree 7.

The above can be generalised as follows: consider dividing $f_1(x)$ by $(x-c)$. We then have

$$f_1(x) = (x-c)Q(x) + R_1.$$

The divisor $(x-c)$ becomes a factor by rewriting the above as

$$f_2(x) = f_1(x) - R_1 = (x-c)Q(x),$$

Where upon $(x-c)$ is a factor of $f_2(x) = f_1(x) - R_1$ implying that $f_2(x)$ is a completely factorisable polynomial.

Continuing this process we have

$$f_2(x) = (x-c)[(x-d)P(x) + R_2],$$

i.e.

$$f_3(x) = (x-c)(x-d)P(x),$$

where $f_3(x)$ is now a completely factorisable polynomial given by $f_3(x) = f_2(x) - (x-c)R_2$.

Recall that, initially, $f_1(x)$ was divided by the divisor $(x-c)$, and this gave us a constant remainder. In having divided by another divisor $(x-d)$ we have effectively divided $f_1(x)$ by a quadratic divisor $(x-c)(x-d)$. We know from the remainder theorem that this gives a linear remainder of the form $Ax+B$, and this is what we have as $(x-c)R_2$. In having brought this remainder over to the right hand side we now have a completely factorised function $f_3(x)$.

Similarly

$$f_3(x) = (x-c)(x-d)[(x-e)T(x)+R_3],$$

implying

$$f_4(x) = (x-c)(x-d)(x-e)T(x),$$

where $f_4(x) = f_3(x) - (x-c)(x-d)R_3$. Now we have divided $f_1(x)$ by a cubic divisor $(x-c)(x-d)(x-e)$ so our remainder is (at most) quadratic remainder. This is indeed what we have as $(x-c)(x-d)R_3$.

This process of continually dividing successive quotients by linear divisors allows us to transform of polynomial $f_1(x)$ into a fully factored form, whereby those divisors now become factors. We therefore transformed $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ into

$$f_n(x) = (x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n)$$

where

$$f_n(x) = f_{n-1}(x) - \underbrace{(x-x_{n-1})(x-x_{n-2})\ldots(x-x_2)}_{n-2 \text{ divisors}}.R_{n-1} \qquad ,\ldots$$

$$\ldots, \qquad f_4(x) = f_3(x) - (x-x_3)(x-x_2)R_3, f_3(x) = f_2(x) - (x-x_2)R_2, f_2(x) = f_1(x) - R_1,$$

where $f_1(x)$ is the result of dividing $f(x)$ by $(x-x_1)$.

<u>*Example:*</u> We want to construct a factorised polynomial from $f(x)$ such that our factorised polynomial contains $(x-1),(x-2),(x-3)$ as factors. So consider dividing $f_1(x) = x^5 + 2x^4 - 3x^3 - x^2 + 3x - 1$ by $(x-1)$, and then dividing the successive quotients by $(x+2)$ and $(x+3)$ respectively. By the remainder theorem we obtain

$$x^5 + 2x^4 - 3x^3 - x^2 + 3x - 1 = (x-1)(x^4+3x^3-x+2)+1.$$

Therefore

$$x^5 + 2x^4 - 3x^3 - x^2 + 3x - 2 = (x-1)(x^4 + 3x^3 - x + 2)$$

and $(x-1)$ is now a factor of $f_2(x) = x^5 + 2x^4 - 3x^3 - x^2 + 3x - 2$. Dividing $x^4 + 3x^3 - x + 2$ by $(x+2)$ gives

$$x^5 + 2x^4 - 3x^3 - x^2 + 3x - 2 = (x-1)[(x-2)(x^3 + x^2 - 2x + 3) - 4].$$

Therefore

$$x^5 + 2x^4 - 3x^3 - x^2 + 3x - 2 - [-4(x-1)] = (x-1)(x-2)(x^3 + x^2 - 2x + 3).$$

and $(x+2)$ is now a factor of $f_3(x) = x^5 + 2x^4 - 3x^3 - x^2 + 7x - 6$. Dividing $x^3 + x^2 - 2x + 3$ by $(x+3)$ gives $f_3(x)$ to be

$$x^5 + 2x^4 - 3x^3 - x^2 + 7x - 6 = (x-1)(x-2)[(x+3)(x^2 - 2x + 4) - 9].$$

Therefore

$$x^5 + 2x^4 - 3x^3 + 8x^2 + 16x + 24 = (x-1)(x-2)(x+3)(x^2 - 2x + 4),$$

so that $(x+3)$ is now a factor of $f_4(x) = x^5 + 2x^4 - 3x^3 + 8x^2 + 16x + 24$. From the initial polynomial $f_1(x)$, having coefficients of our own choosing, we have therefore been able to create a fully factorisable polynomial $f_4(x)$ up to the irreducible quadratic $x^2 - 2x + 4$.

2.8.3 Repeated factors

For this topic you will need to know the product rule for differentiation.

Let us start with the completely factorisible polynomial $f(x) = x^3 - x^2 - x + 1$. We can verify this by testing $f(x)$ for a few values of x: $f(1) = 0$ and $f(-1) = 0$. This means that $(x-1)$ and $(x+1)$ are factors of $f(x)$ and we have

$$f(x) = (x-1)(x+1)P(x).$$

We now need to find $P(x)$, the remaining factor of $f(x)$. it so happens that $(x-1)$ is again a factor of $f(x)$ implying that

$$f(x) = (x-1)^2(x+1).$$

The problem is that we have already tested for $f(1)$. Testing it again won't give us $(x-1)$ as another factor. So how do we find $(x-1)$ to be this repeated factor?

We find it by differentiating $f(x)$ and using the factor theorem again. In that case we obtain

$$f'(x) = 3x^2 - 2x - 1.$$

Then

$$f'(1) = 3(1)^2 - 2(1) - 1 = 0.$$

This allows us to say that $(x-1)$ is not just a single factor but a repeated factor (i.e. a factor appearing more than once). What we have seen above is not a one off, but applies in general.

To see this, suppose that an n^{th} polynomial $f(x)$ contains the repeated factor $(x-c)^2$. In that case $f(x)$ can be written as

$$f(x) = (x-c)^2 P(x)\,, \tag{17}$$

for some polynomial $P(x)$ of degree $n-2$. We need to find a way of testing $f(x)$ such that we know it has a repeated factor. To do this let us differentiate (17). We obtain

$$f'(x) = 2(x-c)P(x) + (x-c)^2 P'(x)$$

which becomes

$$f'(x) = (x-c)[2P(x) + (x-c)P'(x)] \tag{18}$$

Notice now that, by the factor theorem, we have $f'(c) = 0$ showing that $(x-c)$ is a factor of $f'(x)$. Equation (18) is the key equation we need. It allows us to confirm whether or not a factor is a repeated factor of a given polynomial.

Note the use of language: If a polynomial has factor $(x-c)$ and then again the factor $(x-c)$, then the polynomial has a quadratic factor $(x-c)^2$ but the factor $(x-c)$ has been *repeated once* only. Similarly for a cubic factor $(x-c)^3$ the factor $(x-c)$ has been *repeated twice* only, etc.

So we now state the factor theorem for a factor repeated once (i.e. for a polynomial having a squared factor):

Factor theorem for repeated factors: Quadratic case

Let an n^{th} degree polynomial $f(x)$ be divided by $(x-c)$. Then

if $f(c) = f'(c) = 0$ then $(x-c)^2$ is a factor of $f(x)$.

In the example above involving $f(x) = x^3 - x^2 - x + 1$ we are now justified in performing $f'(1) = 0$ and claiming that $(x - 1)$ is a repeated factor of the cubic. Hence

$$f(x) = (x - 1)^2(x + 1).$$

Example 1: To determine whether or not $f(x) = x^4 - 2x^3 - 3x^2 + 8x - 4$ has repeated factors let us first use the factor theorem to find one factor. So, if $x = 2$, we obtain $f(2) = 0$, hence $(x - 2)$ is a factor of $f(x)$. Is it a repeated factor? To find out, differentiate $f'(x)$ to obtain

$$f'(x) = 4x^3 - 6x^2 - 6x + 8.$$

Then $f'(2) = 4$, implying that $(x - 2)$ is not a repeated factor. Testing $f(x)$ when $x = 1$ we obtain $f(1) = 0$ implying $(x - 1)$ is a factor of $f(x)$. Testing $f'(1)$ we find that $f'(1) = 0$ hence we know that $(x - 1)^2$ is a factor of $f(x)$. Therefore we have

$$x^4 - 2x^3 - 3x^2 + 8x - 4 = (x - 2)(x - 1)^2 Q(x)\,,$$

where $Q(x)$ is linear (it is left as an exercise for you to find $Q(x)$ and complete the factorisation).

Example 2: Let us see if $f(x) = x^4 + 2x^3 - 11x^2 - 12x + 36$ has any repeated factors. Testing $f(x)$ for a few values of x we obtain $f(1) = 16$, $f(-1) = 36$, $f(2) = 0$, hence $x = 2$ is a root of $f(x)$ implying $(x - 2)$ is a factor of $f(x)$. Differentiating $f(x)$ we have $f'(x) = 4x^3 + 6x^2 - 22x - 12$. Testing $f'(2)$ we obtain $f'(2) = 0$, therefore $(x - 2)$ is a repeated factor of $f(x)$.

Testing $f(x)$ for other values of x we find that $f(-3) = 0$. Then, substituting $x = 3$ into $f'(x)$ we obtain $f'(-3) = 0$. Hence $(x + 3)$ is also a repeated factor of $f(x)$.

Since $f(x)$ is a quartic and we have found two quadratic factors of $f(x)$ there are no more x values to test. Hence

$$x^4 + 2x^3 - 11x^2 - 12x + 36 = (x - 2)^2(x + 3)^2.$$

Example 3: In this example we will perform arithmetic involving the imaginary number $i = \sqrt{-1}$. So, to determine whether or not $f(x) = x^5 - x^4 + 2x^3 - 2x^2 + x - 1$ has repeated roots, let us test $f(x)$ for various values of x:

- we have $f(-1) = 8$ hence $x = -1$ is not a root of $f(x)$, and $(x + 1)$ is not a factor of $f(x)$.

- we have $f(1) = 0$ hence $x = 1$ is a root of $f(x)$, and $(x-1)$ is a factor of $f(x)$. To test if this factor is a repeated factor we differentiate $f(x)$ to obtain

$$f'(x) = 5x^4 - 4x^3 + 6x^2 - 4x + 1$$

  from which $f'(1) = 4 \neq 0$. Hence $x = 1$ is not a double root, implying $(x-1)$ is not a repeated factor of $f(x)$;

Testing other values of x will not produce any other real roots of $f(x)$. So we are left with two options: we either compare coefficients of $x^5 - x^4 + 2x^3 - 2x^2 + x - 1 = (x-1)Q(x)$, where $Q(x) = (ax^4 + bx^3 + cx^2 + dx + e)$, or we try testing imaginary numbers. It so happens that if $x = \pm i$, then

$$\begin{aligned} f(\pm i) &= (\pm i)^5 - (\pm i)^4 + 2(\pm i)^3 - 2(\pm i)^2 + (\pm i) - 1\,, \\ &= \pm i - 1 + 2(\mp i) + 2 \pm i - 1 = 0. \end{aligned}$$

Therefore, $x = \pm i$ are roots of $f(x)$. From this we obtain $x^2 = -1$, implying that $x^2 + 1$ is a factor of $f(x)$. Testing $f'(\pm i)$ we have

$$f'(\pm i) = 5(\pm i)^4 - 4(\pm i)^3 + 6(\pm i)^2 - 4(\pm i) + 1 = 5 \pm 4i - 6 \mp 4i + 1 = 0$$

implying that $x^2 + 1$ is a repeated factor. Hence

$$x^5 - x^4 + 2x^3 - 2x^2 + x - 1 = (x-1)(x^2+1)^2.$$

Note that we need both conditions of $f(c) = 0$ **and** $f'(c) = 0$ for us to know whether or not a factor is repeated. To see why this is so notice that both $(x-c)^2$ and $(x-c)(x-d)$ are quadratic factors. In this last case a polynomial $f(x)$ can be written as

$$f(x) = (x-c)(x-d)Q(x) \qquad [*]$$

from which

$$f'(x) = (x-d)Q(x) + (x-c)Q(x) + (x-c)(x-d)Q'(x). \qquad [**]$$

Substituting $x = c$ into [*] gives $f(c) = 0$ implying $(x-c)$ is a factor of $f(x)$. Substituting $x = c$ into [**] gives $f'(c) = (c-d)Q(c) \neq 0$ implying $(x-c)$ is not a repeated factor of $f(x)$.

2.8.4 *Extending the idea of repeated factors*

We saw in the previous section that if a polynomial $f(x)$ of degree n has a repeated factor $(x-c)$ then both $f(c)=0$ **and** $f'(c)=0$. But this test only works up to quadratic factors. What if $f(x)$ has a factor $(x-c)^3$ or $(x-c)^4$? What if $f(x)$ has a repeated factor $(x-c)^m$ for $m \leq n$? Another way of putting this is, How do we find out how many times a factor is repeated?

Well, let us consider how we can use the factor theorem to test for a factor $(x-c)^3$. In this case, by the factor theorem we have

$$f(x) = (x-c)^3 P(x). \tag{19}$$

Whereas in the previous section we needed to differentiate once to get the necessary second condition for testing for repeated factors, here we are going to have to differentiate twice to get a necessary third condition for testing whether or not a factor is a cubic factor.

Hence

$$\begin{aligned} f'(x) &= 3(x-c)^2P(x) + (x-c)^3P'(x)\,, \\ f''(x) &= 6(x-c)P(x) + 6(x-c)^2P'(x) + (x-c)^3P''(x)\,, \\ &= (x-c)[6P(x) + 6(x-c)P'(x) + (x-c)^2P''(x)]\,. \end{aligned} \tag{20}$$

Notice that $(x-c)$ is now a factor not only of $f'(x)$ but also of $f''(x)$. Therefore our third necessary condition for testing whether or not $(x-c)$ is a cubic factor if given by $f''(c)=0$. We can now state the factor theorem for a cubic factor:

Factor theorem for repeated factors: Cubic case

Let an n^{th} degree polynomial $f(x)$ be divided by $(x-c)$. Then

if $f(c) = f'(c) = f'''(c) = 0$ then $(x-c)^3$ is a factor of $f(x)$.

<u>*Example 1:*</u> Let us test $f(x) = x^5 + 7x^4 + 19x^3 + 25x^2 + 16x + 4$ for repeated factors. By the factor theorem we can test various values of x as follows: $f(1) = 72$, $f(2) = 432$, $f(-1) = 0$. Hence $(x+1)$ is a factor of $f(x)$.

Differentiation $f(x)$ twice we obtain

$$f'(x) = 5x^4 + 28x^3 + 57x^2 + 50x + 16$$

and

$$f''(x) = 20x^3 + 84x^2 + 114x + 50.$$

We now test $f'(x)$ and $f''(x)$ with $x = -1$: $f'(-1) = 5 - 28 + 57 - 50 + 16 = 0$, and $f''(-1) = -20 + 84 - 114 + 50 = 0$. Hence $(x+1)$ is a repeated factor, and is in fact a cubic factor.

Since $f(x)$ is a quintic, and we have found a cubic factor, there remains a quadratic factor to be found. Testing again some x values in $f(x)$ we find $f(-2) = 0$. Then $f'(-2) = 5(16) - 28(8) + 57(2) + 100 + 16 = 0$ implying that $f(x)$ has factor $(x+2)^2$.

Note: Having found $(x+1)^3$ as a factor how do we know that there isn't another $(x+1)$ factor? Well, we can differentiate $f(x)$ a third time to see if $f'''(-1)$ equals zero. In this case

$$f'''(x) = 60x^2 + 168x + 114,$$

from which $f'''(-1) = 6 \neq 0$. Hence $(x+1)$ is a factor only up to cubic order.

Exercise: Find the linear, quadratic and cubic factors of

$$f(x) = 24x^7 + 116x^6 + 122x^5 - 125x^4 - 160x^3 + 88x^2 + 32x - 16.$$

We should now be able to see a pattern emerging: if $(x-c)^m$ is a factor of $f(x)$ where c is a real number, then

$$\boxed{f(c) = f'(c) = f''(c) = \cdots = f^{(m-1)}(c) = 0}$$

where $f^{(m-1)}(c)$ is the $(m-1)^{th}$ derivative of $f(x)$ evaluated at $x = c$. This is the test we use to determine how many times a factor is repeated.

More generally, when an n^{th} degree polynomial $f(x)$ is divided by $(x-x_1)^{m_1}, (x-x_2)^{m_2}, \ldots, (x-x_k)^{m_k}$, where $k \leq n$, the factor theorem for repeated factors says that:

- when $f(x_1) = f'(x_1) = f''(x_2) = \cdots = f^{(m_1-1)}(x_1) = 0$, $(x-x_1)^{m_1}$ is a factor of $f(x)$;
- when $f(x_2) = f'(x_2) = f''(x_2) = \cdots = f^{(m_2-1)}(x_2) = 0$, $(x-x_2)^{m_2}$ is a factor of $f(x)$;

- ...
- when $f(x_k) = f'(x_k) = f''(x_k) = \cdots = f^{(m_k-1)}(x_k) = 0$, $(x - x_k)^{m_k}$ is a factor of $f(x)$.

and

$$f(x) = (x - x_1)^{m_1}\,(x - x_2)^{m_2}\ \ldots (x - x_k)^{m_k} Q(x),$$

for some $Q(x)$ of degree $n - (m_1 + m_2 + \cdots + m_k)$.

Example 2: Let us test $f(x) = 24x^7 + 116x^6 + 122x^5 - 125x^4 - 160x^3 + 88x^2 + 32x - 16$. By trial and error we find that $f(1/2) = 0$ so that $(2x - 1)$ is a factor of $f(x)$. We now differentiate $f(x)$ and evaluate this derivative at $x = 1/2$ to see if $2x - 1$ is a repeated factor:

$$f'(x) = 168x^6 + 696x^5 + 610x^4 - 500x^3 - 480x^2 + 176x + 32 \quad \Rightarrow \quad f'\left(\frac{1}{2}\right) = 0,$$

$$f''(x) = 1008x^5 + 3480x^4 + 2440x^3 - 1500x^2 - 960x + 176 \quad \Rightarrow \quad f''\left(\frac{1}{2}\right) = 125$$

So $2x - 1$ is a quadratic factor only, and we have

$$f(x) = (2x - 1)^2 Q(x).$$

Looking for another roots of $f(x)$ we find $f(-2) = 0$. Hence $x + 2$ is a factor of $f(x)$. Testing for repeated factors we find that $f'(-2) = f''(-2) = 0$ but that $f'''(-2) = 3600$ (left as an exercise to verify). So $(x + 2)$ is a cubic factor of $f(x)$ and we have

$$f(x) = (2x - 1)^2 (x + 2)^3 P(x).$$

We have factored out of $f(x)$ a polynomial of degree 5. Since $f(x)$ is of degree 7, $P(x)$ is a quadratic. By long division or comparing coefficients we find that $P(x) = 6x^2 - x - 2$ (left as an exercise). Since $P(x) = (2x + 1)(3x - 2)$ is not a perfect square there are no more repeated factors, and we have

$$f(x) = (x + 2)^3 (2x - 1)^2 (2x + 1)(3x - 2).$$

2.8.5 The rational root theorem

The approach of using educated guesswork (more properly, the factor theorem) to find factors of a polynomial $f(x)$ is very hit-and-miss. More than this it also very limited. For example, if $f(x)$ has a factor $(7x - 13)$ how many of us would thing of guessing $x = 13/7$ as a possible root? Also, if we had a polynomial of degree 7 how easy would it be to guess a root for that polynomial? The question therefore is, Is there a method which can help us factorise a polynomial of any degree more systematically and easily? The answer is yes, but with certain limitations. This method is called the *rational roots test*. The rational root test is designed to use the leading coefficient a_n and the constant term a_0 of a polynomial $f(x)$ to help find rational roots $x = p/q$ of $f(x)$.

To see this consider the following polynomial

$$f(x) = (2x - 7)(5x + 1)(3x + 11) = 30x^3 + 11x^2 - 384x - 77 = 0.$$

It seems clear that it would be well nigh impossible to guess the factors as shown. Now consider

- the roots of $f(x)$. These are $7/2, -1/5, -11/3$.
- the factors of the leading coefficient, these being $\pm 2, \pm 3$ and ± 5 (other factors of 30 include ± 1, ± 10, ± 15, and ± 30. None of these numbers are the coefficients of x in the factored version of $f(x)$, but we won't know this at the beginning. We will only know the expanded form of $f(x)$).
- the factors of the costant term, these being $\pm 7, \pm 1$, and ± 11 (other factors of the constant terms include ± 77 but this does not form part of the roots of x. Again we won't know this at the start of our factorisation process).

So when we go through complete examples on factoring a polynomial in this way we will in fact have to list all the possible factors (positive and negative) of the leading term a_n and of the constant term a_0. Then in that list, somewhere, will be the factors leading to the actual roots x, and therefore leading to the linear factors of $f(x)$.

Thus, more generally if we consider $f(x) = (px - q)(ax - b) = 0$ then the roots are $x = q/p$ and $x = b/a$. But

$$(px - q)(ax - b) = pax^2 - x(pb + qa) + qb\ .$$

Here we see that the coefficient of x^2 has factors p and a, and the constant has factors q and b, precisely the numbers which make up the roots/factors.

From this we can develop a procedure for finding the rational roots (which also includes integer roots) of an n^{th} degree polynomial as follows: given

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

we find the factors $\pm p$ of a_0, and $\pm q$ of a_0 respectively. We then test various combinations of $\pm p/q$ to determine which give $f(\pm p/q) = 0$.

For example, if $f(x) = 6x^6 - 23x^5 - 36x^4 + 219x^3 - 170x^2 - 108x + 72$ we consider factors of 6 to be $\pm\{1, 2, 3, 6\}$ and factors of 72 to be $\pm\{1, 2, 3, 6, 12, 24, 72\}$. From these two sets of factors we can set up fractions $\pm p/q$ and test $f(\pm p/q)$. If this equals zero then $(qx \mp p)$ is a factor of $f(x)$.

Hence, let us test $x = 1/2$ in $f(x)$. Then $f(1/2) = 0$ which implies that $x = 1/2$ is a root of $f(x)$, and $(2x - 1)$ is a factor of $f(x)$. Let us now test $x = -1/3$.

Then $f(-1/3) = 19600/243$, so $x = -1/3$ is not a root of $f(x)$. And again: $x = -2/3$ gives $f(-2/3) = 0$, hence $x = -2/3$ is a root of $f(x)$, and $(3x + 2)$ is a factor of $f(x)$. And so on.

The thing to notice here is that we have significantly reduced our effort in finding roots and factors. It is no longer a matter of total guesswork, but more a matter of testing certain combinations of factors from a limited set of factors. This is the *rational root test*, and can be generalized as the *rational root theorem*.

<u>*Example 1:*</u> We want to use the rational root test to factorise

$$f(x) = 18x^4 - 27x^3 + x^2 + 12x - 4.$$

Let p and q be the factors of the constant term and the leading term respectively. Then we have

$$p = \pm\{1, 2, 4\} \text{ and } q = \pm\{1, 2, 3, 6, 9, 18\}.$$

Let us now test as follows:

$x = p/q$	$1/1$	$-1/1$	$1/2$	$-1/2$	$1/3$	$-1/3$
$f(x)$	0	30	0	$-21/4$	$-2/3$	$-20/3$

$x = p/q$	$2/1$	$-2/1$	$2/3$	$-2/3$
$f(x)$	96	480	0	0

A quartic has four roots, and we have found four zeros, so we can stop. Therefore

$$f(x) = (x-1)(2x-1)(3x-2)(3x+2).$$

Example 2: Let us use the rational root test to factorise

$$f(x) = 12x^4 - 40x^3 - 5x^2 + 45x + 18,$$

and let p and q be factors of the constant term and leading coefficient respectively. Then we have

$$p = \pm\{1, 2, 3, 9, 18\} \text{ and } q = \pm\{1, 2, 3, 4, 6, 12\}.$$

Testing a few combinations of p/q we see that $f(1) = 30$, $f(-1) = 20$, $f(1/2) = 35$, $f(-1/2) = 0$. From this last test we see that $(2x+1)$ is a factor of $f(x)$. Continuing, we have $f(2) = -40$, $f(-2) = 420$, $f(2/3) = 980/27$, $f(-2/3) = 0$. Hence $(3x+2)$ is another factor of $f(x)$.

Since $f(x)$ is quartic and we have found two factors of $f(x)$ we can write

$$12x^4 - 40x^3 - 5x^2 + 45x + 18 = (2x+1)(3x+2)(ax^2 + bx + c).$$

From this we can expand and compare coefficients (left as an aexercise) to show that $a = 2$, $b = -9$, and $c = 9$. Hence

$$12x^4 - 40x^3 - 5x^2 + 45x + 18 = (2x+1)(3x+2)(2x^2 - 9x + 9).$$

and the complete factorisation of $f(x)$ is

$$12x^4 - 40x^3 - 5x^2 + 45x + 18 = (2x+1)(3x+2)(x-3)(2x-3).$$

Example 3: Let us use the rational root test to factorise

$$f(x) = 4x^5 - 37x^3 + 8x^2 + 75x - 50.$$

Again, let p and q be factors of the constant term and leading coefficient respectively. Then we have

$$p = \pm\{1, 2, 5, 10, 25, 50\} \text{ and } q = \pm\{1, 2, 4\}.$$

Setting up a table of values for $x = p/q$ can systematically test various combination of p/q:

$x = p/q$	$1/4$	$1/2$	1	2	5	$5/2$	10
$f(x)$	$-\frac{8019}{256}$	-15	0	-36	8400	0	364500

So by testing positive values of p/q we have found $(x - 1)$ and $(2x - 5)$ to be factors of $f(x)$. Since $f(10)$ is a large number it is pointless testing $f(25)$ and $f(50)$ so let us now test for negative values of p/q:

$x = p/q$	-5	$-5/2$	-2
$f(x)$	-8100	0	0

From the table of values above we see that $(2x + 5)$ and $(x + 2)$ are factors of $f(x)$. Since $f(x)$ is a quintic and we have found a total of four factors, we can write

$$4x^5 - 37x^3 + 8x^2 + 75x - 50 = (2x + 5)(x + 2)(x - 1)(2x - 5)(ax + b),$$

And find a and b by comparing coefficients (or again by testing more values of p and q). This is left as an exercise, but ultimately we obtain $a = 1$ and $b = -1$. Hence the complete factorisation of $f(x)$ is

$$4x^5 - 37x^3 + 8x^2 + 75x - 50 = (2x + 5)(x + 2)(2x - 5)(x - 1)^2.$$

In summary we can state the *rational root theorem* as follows: Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0 \tag{21}$$

be a polynomial with integer coefficients, where $n \in \mathbb{N}$. If p and q are two integers which are coprime (i.e. integers not having any common factors), and p/q is a root of $f(x)$, then p is a factor of a_0 and q is a factor of a_n.

<u>Proof:</u> We prove this in two stages. One stage is to isolate the term involving a_n, and the other stage is to isolate the terms involving a_0. But first we substitute $x = p/q$ into $f(x)$ to obtain

$$a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + a_{n-2}\left(\frac{p}{q}\right)^{n-2} + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0$$

Multiplying across by q^n we have

$$a_np^n + a_{n-1}p^{n-1}q + a_{n-2}p^{n-2}q^2 \ldots + a_1pq^{n-1} + a_0q^n = 0\ ,$$

$$\therefore \qquad a_np^n + a_{n-1}p^{n-1}q + a_{n-2}p^{n-2}q^2 \ldots + a_1pq^{n-1} = -a_0q^n. \qquad [*]$$

We can factorise p in the left hand side to give

$$p(a_np^{n-1} + a_{n-1}p^{n-2}q + a_{n-2}p^{n-3}q^2 \ldots + a_1q^{n-1}) = -a_0q^n \qquad (22)$$

This is of the form $p.k = -a_0q^n$. Since p and q are co-prime, p is a factor of a_0.

The same analysis as above can be used to show that q is a factor of a_n, but this time we would keep a_0q on the left hand side of [*] and move a_np^n to the right hand side of [*].

We would then factorise q to obtain an expression of the form $qk = a_np^n$. And again, since p and q are co-prime, q is a factor of a_n . ■

Now, to understand that $p.k = -a_0q^n$ implies p is a factor of a_0 consider divisibility being defined indirectly via multiplication. As such we can say that a number n is divisible by m if $n = mq$ for some integer q. In other words, an integer multiple of m will give n, and this is equivalent to saying that n is divisible by m. So, if n is divisible by 6 then $n = 6q$ for some integer q. Since $n = 2(3q)$ we can also say that n is divisible by 2.

Extending this idea consider the different ways in which the factors of the number 108 can be combined to make 108:

- $2 \times 54 = 3 \times 36$. The number 2 doesn't divide 3 but it does divide 36. Hence, because 2 and 3 are co-prime 2 is a factor of 36;
- $2 \times 54 = 9 \times 12$. Then number 2 doesn't divide 9 but it does divide 12. Hence, because 2 and 9 are co-prime, 2 is a factor of 12.

So, in general, if p and q are co-prime, p is a factor of a_0. Ditto for the case that q is a factor of a_n.

What the rational root theorem allows for is a finite set of values (the factors of p and q) for which $x = p/q$ may be a root. Notice (as previously mentioned) that not all combinations of factors of p and q will form the valid rational root. Hence we need to test each formation $x = p/q$ separately (including negative versions).

2.8.6 The rational root test in conjunction with the intermediate value theorem

Note that we don't necessarily need to test all the possible combinations of p/q. In practice it we need only test as many combinations as we need to reduce our polynomial to a cubic or quadratic. Then we can use our usual approaches to factorising cubics and quadratics.

For example, if we had a polynomial such as $f(x) = 324x^4 - 279x^3 - 117x^2 + 124x - 12$ our set of p and q values would be

$$p = \pm\{1, 2, 3, 6\}$$

$$q = \pm\{1, 2, 3, 4, 6, 9, 12, 18, 27, 36, 54, 81, 108, 162, 324\}$$

There are a lot of values of p/q to test here. But all we need do is find two roots (say $x = x_1$ and $x = x_2$), after which we could find the remaining two roots by solving $f(x) = (x - x_1)(x - x_2)(ax^2 + bx + c)$ in the usual way.

Even with this approach there is a (possibly) quicker, or more systematic, way in which we can factorise polynomials having large values for leading coefficients and constants, especially when our polynomial is of a high degree.

So, consider again the quartic above. By testing some values of $x = p/q$ we find

$$f(1/2) = 49/8 \text{ and } f(-1/2) = -385/8.$$

Now, notice something about these two values: one is positive and one is negative. This means that somewhere in between $x = 1/2$ and $x = -1/2$ there must be a value $x = c$ where $f(c) = 0$, i.e there is a value of x in $[-1/2, 1/2]$ which is a root of $f(x)$. Knowing this helps us choose a better combination of p and q from the sets above. The only possible option is to test $x = \pm 1/3, \pm 1/4 \pm 1/6$, etc. Doing this we find that

$$f(1/9) = 0,$$

implying that $(9x - 1)$ is a factor of $f(x)$. Continuing to test various combinations of p/q way we may end up finding

$$f(-1/3) = -52 \text{ and } f(1) = 40.$$

Since one value is negative and the other is positive there must be a value of x, say $x = c$, in $[-1/3, 1]$ which gives us $f(c) = 0$. We have already tested $x = 1/9$ so let us test $x = 2/3$. This gives

$$f(2/3) = 0,$$

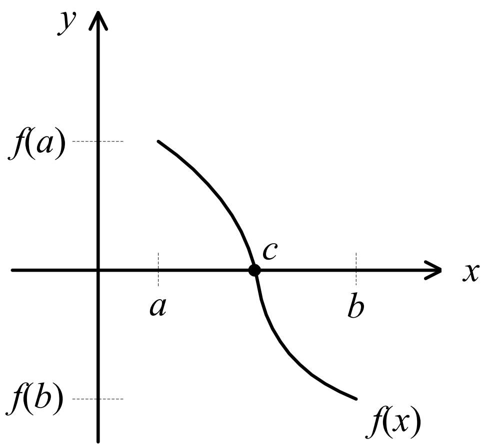
implying that $(3x - 2)$ is a factor of $f(x)$. So we have

$$f(x) = (9x - 1)(3x - 2)Q(x),$$

after which we can solve for $Q(x)$ either by continuing to test rational roots or by using other usual methods.

It should be noted that we can't "engineer" the situation where we get function values of opposite sign. This tends to happen accidentally during our process of testing rational fractions. But if we do come across a situation where we have found function values of opposite sign then we can immediately use the idea above to reduce the amount of testing we need to go through.

The idea that two values of $f(x)$ having signs opposite to each other implies a value of x such that $f(x) = 0$ is formalised as the *Intermediate Value Theorem*. This situation is informally illustrated as seen below:

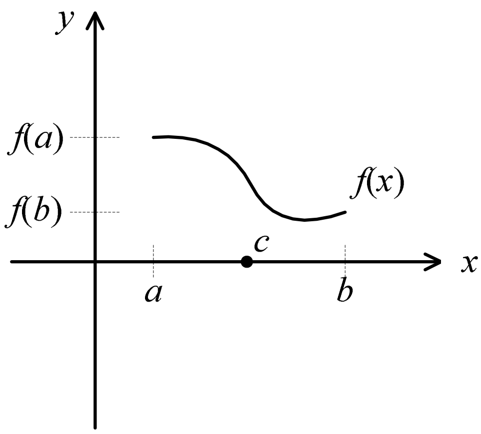


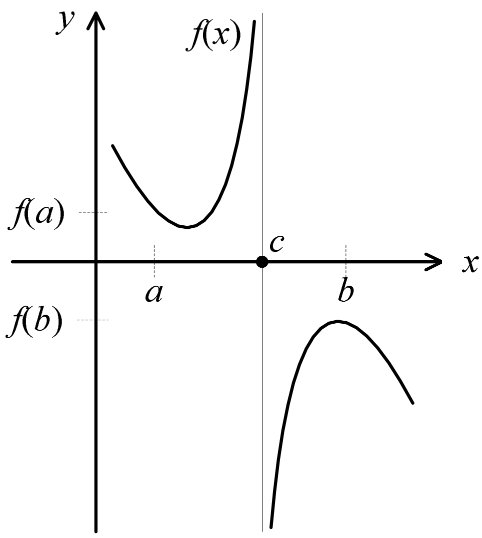
Because $f(x)$ is continuous we see that there is a value $x = c$ between $x = a$ and $x = b$ such that $f(c) = 0$, implying that $x = c$ is a root of $f(x)$, and that $(x - c)$ is a factor of $f(x)$.

So we can formally state the *Intermediate Value Theorem* as

> Let f be a continuous function on the interval $[a, b]$. If $f(a)$ and $f(b)$ have opposite signs then there is some value $c \in (a, b)$ such that $f(c) = 0$

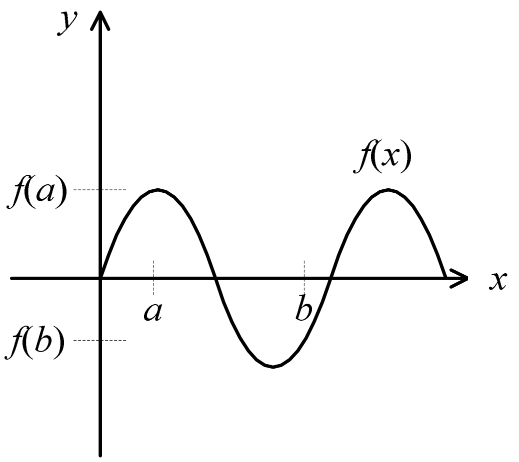
Note that $f(x)$ has to be continuous and $f(a)$ and $f(b)$ have to be of opposite signs. If not we may end up with the situations illustrated below:

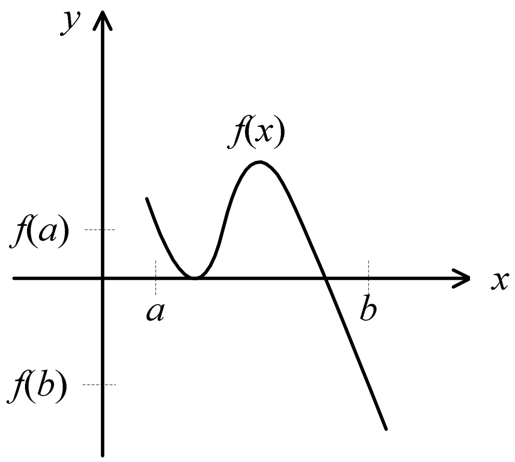




In the left diagram the function values do not have opposite signs, therefore having no roots in $[a, b]$, and in the right diagram the function is not continuus at $x = c$.

Also note that the intermediate value theorem guarantees at least one root. But there may be more than one root in $[a, b]$, or the root may be a repeated root:





Example 1: As an example of using the intermediate value theorem as part of the rational root test consider factorising

$$f(x) = 48x^4 - 116x^3 - 224x^2 + 129x + 63.$$

Let p be the set of factors of the constant term, and let q be the set of factors of the leading coefficient. Then we have

$$p: \pm\{1, 2, 3, 7, 9, 21, 63\}$$

$$q: \pm\{1, 2, 3, 4, 6, 8, 12, 16, 24, 48\}.$$

Then $x = p/q$, for various values of p and q, can be tested, as shown below:

$x = p/q$	-1	$-1/2$	$1/2$	1	$3/2$
$f(x)$	-126	-40	60	-100	-396

Here we see that $f(x)$ goes from negative to positive for $x \in [-1/2, 1/2]$, and then goes from positive to negative for $x \in [-1/2, 1]$.

Therefore let us test $f(x)$ when $x = -1/3$, and $x = 3/4$:

$$f(-1/3) = 0 \text{ and } f(3/4) = 0.$$

Hence $(3x + 1)$ and $(4x - 3)$ are factors of $f(x)$.

Since we have found two of the four factors we can find the remaining two factors by setting up a quadratic and comparing coefficients (we could, or course, continue testing various values of p/q). Hence

$$48x^4 - 116x^3 - 224x^2 + 129x + 63 = (4x - 3)(3x + 1)(ax^2 + bx + c).$$

Multiplying out the right hand side and collecting terms in powers of x gives

$$48x^4 - 116x^3 - 224x^2 + 129x + 63$$
$$= 12ax^4 + x^3(12b - 5a) + x^2(12c - 5b - 3a) + x(-5c - 3b) - 3c.$$

Comparing coefficient we have

$$\begin{aligned} 48 &= 12a \\ -116 &= 12b - 5a \\ -224 &= 12c - 5b - 3a \\ 129 &= -5c - 3b \\ 63 &= -3c \end{aligned}$$

Solving appropriately gives

$$a = 4, b = -8, c = -21.$$

Hence $f(x)$ is

$$48x^4 - 116x^3 - 224x^2 + 129x + 63 = (4x - 3)(3x + 1)(4x^2 - 8x - 21).$$

Either by the quadratic formula, or by the factor theorem we can factorise the quadratic (left as an exercise) to give the complete factorisation of $f(x)$ to be

$$48x^4 - 116x^3 - 224x^2 + 129x + 63 = (4x - 3)(3x + 1)(2x - 7)(2x + 3).$$

Example 2: Let use the rational root test and the intermediate value theorem to factorise $f(x) = 14x^4 - 81x^3 + 162x^2 - 124x + 24$. Here the set of factor of $a_0 = 24$ and $a_4 = 14$ are respectively

$$p: \pm\{1,2,3,4,6,8,12,24\} \text{ and } q: \pm\{1,2,7\}.$$

Then $x = p/q$, for various values of p and q, can be tested, as shown below:

$x = p/q$	$1/2$	1	$12/7$	2
$f(x)$	$-27/4$	-5	$120/343$	0

Here we see that $x = 2$ is a root, implying $(x - 2)$ is a factor of $f(x)$. We also see a change of sign between $x = 1$ and $x = 12/7$ implying that $f(x)$ has a root when x is between these two values. Knowing this, let us therefore test $f(x)$ when $x = 3/2$: $f(3/2) = 0$ implying $(2x - 3)$ is a factor of $f(x)$. Hence

$$14x^4 - 81x^3 + 162x^2 - 124x + 24 = (x - 2)(2x - 3)Q(x).$$

Once again, having found two of four factors we can set $Q(x) = ax^2 + bx + c$ in the above equation, expand the right hand side, and compare coefficients. Doing this (left as an exercise) we obtain the complete factorisation of $f(x)$ to be

$$14x^4 - 81x^3 + 162x^2 - 124x + 24 = (2x - 3)(7x - 2)(x - 2)^2.$$

You might have thought that it would have been quicker to continue testing values of p/q. However, notice that $(x - 2)$ is a repeated factor. This would not have been found by the rational root test. So we either use the repeated factors work of section 2.8.3, or we compare coefficients as above (although this becomes laborious the more repeated factors there are).

2.8.7 Caveats to the rational root theorem

Caveat 1: There is an important caveat about the rational root theorem which relates to p and q having to be co-prime. For example, if $f(x) = 4x^3 - 13x^2 - 13x + 4$ then the set of factors p of the constant term are $\pm\{1,2,4\}$ and the set of factors q of the leading coefficient are $\pm\{1,2,4\}$. From this let us choose

$$x = \frac{p}{q} = \frac{1}{4}.$$

Testing $x = 1/4$ we have $f(1/4) = 0$ hence $(4x - 1)$ is a factor of $f(x)$.

Let us now say that

$$x = \frac{3}{12}, \qquad \text{or} \qquad x = \frac{5}{20}, \qquad \text{or} \qquad x = \frac{6}{24}, \qquad \dots$$

What we are really saying here is that $x = (kp)/(kq)$ for some integer k. But even though

$$\frac{1}{4} = \frac{3}{12} = \frac{5}{20} = \frac{6}{24} = \cdots,$$

and $f(3/12) = f(5/20) = f(6/24) = 0$, etc., each of the above cases shows that $p = 3, 5$, and 6 do not divide $a_0 = 4$, and $q = 12, 20$ and 24 do not divide $a_3 = 4$. This shows that p/q has to be in its lowest form, i.e. that p and q have to be co-prime.

Caveat 2: This next caveat relates to the fact that the rational root test only works when all the coefficients of the polynomial are integers. For example, suppose we want to find the roots of

$$2x^2 - \frac{9}{2}x + 1 = 0\,. \tag{*}$$

This equation has the same roots as

$$4x^2 - 9x + 2 = 0\,. \tag{**}$$

namely, ¼ and 2.

Applying the rational root test to (*) we see that the factor of a_0 and a_n are respectively, p: $\pm\{1\}$ and q: $\pm\{1, 2\}$. So the possible roots of (*) are ± 1, $\pm$½, or ± 2. In these cases p and q are co-prime but we have lost the root ¼. However, if we consider equation (**) we have factors p: $\pm\{1, 2\}$ and q: $\pm\{1, 2, 4\}$. Now we see that these two sets contain all the necessary integers, i.e. 1, 2, and 4, to form all the correct roots.

The reason for the problem is that it is the dividing of (**) by 2 that has caused us to loose one of the roots. The division has "invisibly" factored out one of the roots. Hence the need to work with polynomials with integer coefficients, not with polynomials with rational coefficients.

On a separate, but related, note consider the case where p is 2 and q is 1. Then we have a possible root $x = 2/1$. It looks here as if p and q not co-prime (since 1 does go into 2 twice). However, this is not the case. For reasons relating to number theory, the integer 1 is not a prime number, so the question of being co-prime does not arise. Even if 1 were to be considered a prime number, 2/1 does not simplify to a fraction in lower terms. In other words, the arithmetic $2/1 = 2$ is not a simplification due to the cancelling of common factors.

So when we say that p/q has to be in lowest terms we mean that p and q must not have common factors, i.e $p/q \neq (km)/(kn)$ where k is a common factor.

Caveat 3: The rational root test only finds rational roots not irrational roots. So if a polynomial has not rational roots the rational root test does not work. For example, consider wanting to use the rational root test to find the roots of

$$f(x) = 2x^2 - 9x + 1 = 0\,.$$

Here we have $p\colon \pm\{1\}$ and $q\colon \pm\{1, 2\}$. Hence the possible roots are ±1, ±2 and/or ±½ . Testing each of these values in the above quadratic shows that $f(x) \neq 0$. Hence none of these rational values are roots, implying that both roots are irrational.

Caveat 4: The rational root test is designed to use the leading coefficient a_n and the constant term a_0 of a polynomial $f(x)$ to help find rational roots $x = p/q$ of $f(x)$. In that case why not simply prove the rational root theorem by factoring $(x - p/q)$ from $f(x)$, and then comparing left hand side and right hand side in the usual way. In other words why not do

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = (x - p/q)(b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0)?$$

Answer: Because this will not work in general to prove what we want. To see why let us complete this process.

$$\begin{aligned} a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 &= \frac{1}{q}(qx - p)(b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0)\,, \\ &= (qx - p)\left(\frac{b_{n-1}}{q}x^{n-1} + \frac{b_{n-2}}{q}x^{n-2} + \cdots + \frac{b_1}{q}x + \frac{b_0}{q}\right), \\ &= q\frac{b_{n-1}}{q}x^n + \cdots - p\frac{b_0}{q}\,. \end{aligned}$$

This seems to say that p is a factor of a_0 and q is a factor of a_n. However, coefficients $q\,\frac{b_{n-1}}{q}$ and $-p\,\frac{b_0}{q}$ are not necessarily integers. For example, we might have

$$a_n = 6 \text{ and } q = 7 \text{ and } b_{n-1}/q = 6/7.$$

This satisfies $a_n = q.\,b_{n-1}/q$ but q is not a factor of a_n.

2.8.8 The rational root test in conjunction with synthetic division by quadratic divisors

This section is based on, and extended from, "Using Synthetic Division By Quadratics To Find Rational Roots", Margaret W. Hutchinson, *The Mathematics Teacher*, Vol. 64, No. 4 (APRIL 1971), pp. 349-352

In the rational root test we construct two sets p and q containing both the positive and negative factors of the constant and leading coefficient of $f(x)$. Then we test each combination $x = +p/q$ and $x = -p/q$ separately. In this section we will find a way of performing the rational root test such that we can test both negative and positive p/q at the same time. During this process we will make use of long division. Specifically we will use the synthetic division process outlined in section 2.6.12.

To illustrate how to test $x = \pm p/q$ in one go consider the following example. Let us factorise

$$f(x) = 3x^3 - 2x^2 - 5x - 6.$$

The sets of factors of the constant and leading coefficients are respectively

$$p\colon \pm\{1, 2, 3, 6\} \text{ and } q\colon \pm\{1, 3\}.$$

The possible roots of $f(x)$ are $\pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{3}, \pm\frac{2}{3}$. Let us test both values of $x = \pm 2$ at the same time. To do this we set up a quadratic based on these values, i.e. $g(x) = (x-2)(x+2) = x^2 - 4$, which we will use as the divisor of $f(x)$.

Dividing $f(x)$ by $g(x)$ we have

					$f(x)$	
			x^3	x^2	x	const
$g(x)$	$-x$	0	3	–2	–5	–6
	–const	4			12	–8
				0	0	
			3	–2	7	–14
		$Q(x)$:	x	const	A	B

where A and B are the coefficients in the remainder $R(x) = Ax + B$. So we have

$$f(x) = (x^2 - 4)(3x - 2) + 7x - 14.$$

When $x = 2$ we have

$$f(2) = (4 - 4)(3(2) - 2) + 7(2) - 14 = 0, \qquad (*)$$

and when $x = -2$ we have

$$f(-2) = (4 - 4)(3(-2) - 2) + 7(-2) - 14 = -28. \qquad (**)$$

So $x = 2$ is a root of $f(x)$, implying that $x - 2$ is a factor of $f(x)$, but $x = -2$ is not a root of $f(x)$ implying $x + 2$ is not a factor of $f(x)$.

Let us now look more closely at equations (*) and (**): notice that $x = 2$ makes both $g(x) = 0$ and $R(x) = 0$, but $x = -2$ only makes $g(x) = 0$. This is what we need in order to determine which of $x = \pm 2$ are roots, namely

if $x = \pm p/q$ makes both $g(x) = 0$ and $R(x) = 0$
then such x is/are roots of $f(x)$.

In practice we will not even have to set up equation (*) and (**). We will in fact be able to test the remainder $R(x)$ at $x = \pm p/q$ and compare the result with the roots of the divisor.

More specifically,

1. Let p and q be the set of factors of the constant term and the leading term of a polynomial $f(x)$;
2. Set up $x = \pm p/q$ as two possible roots of $f(x)$;
3. Now, set up the quadratic divisor $g(x) = (qx - p)(qx + p)$;
4. Divide $f(x)$ by $g(x)$ to obtain $f(x) = (qx - p)(qx + p)Q(x) + Ax + B$;
5. Let $r = -B/A$, where $a \neq 0$. If r equals one of the roots of $g(x)$ then r is a root of $f(x)$;
6. If $A = B = 0$ then both $x = -p/q$ and $x = p/q$ are roots of $f(x)$.

Another example: let use the above idea to factorise $f(x) = 6x^4 - 5x^3 - 12x^2 + 5x + 6$. The set of factors of the constant and leading term are respectively

$$p: \pm\{1, 2, 3, 6\} \text{ and } q: \pm\{1,2, 3, 6\}.$$

Let us test $x = \pm 1$. Then we set up the divisor $\mathrm{g}(x) = (x - 1)(x + 1) = x^2 - 1$. By synthetic division we find $f(x)/g(x)$:

					$f(x)$		
			x^4	x^3	x^2	x	const
$g(x)$	$-x$	0	6	–5	–12	5	6
	–const	1			6	–5	–6
				0	0	0	
			6	–5	–6	0	0
		$Q(x)$:	x^2	x	const	A	B

So $R(x) = 0$ implying that $x = \pm 1$ are both roots of $f(x)$, hence $(x-1)(x+1)$ is a factor of $f(x)$. In this case there is no need to test for $r = -B/A$ (in fact, this is not even possible).

Continuing, let us test $x = \pm 3$. Then $g(x) = (x-3)(x+3) = x^2 - 9$. Dividing $f(x)$ by this $g(x)$ via synthetic division we have

					$f(x)$		
			x^4	x^3	x^2	x	const
$g(x)$	$-x$	0	6	–5	–12	5	6
	–const	9			54	–45	378
				0	0	0	
			6	–5	42	–40	384
		$Q(x)$:	x^2	x	const	A	B

So $R(x) = -40x + 384$. Therefore $r = -B/A = 384/40 = 9\frac{3}{5}$. Since $9\frac{3}{5} \neq \pm 3$ we find that $x = \pm 3$ are not roots of $f(x)$.

Other roots we can test include $x = \pm 1/3$, $x = \pm 2/3$, etc. These cases lead to us setting $g(x) = (3x-1)(3x+1) = 9x^2 - 1$, and $g(x) = (3x-2)(3x+2) = 9x^2 - 4$, in other words quadratic divisors with leading coefficient not equal to 1. If we want to use the usual long division process then this form of $g(x)$ presents no problem, and we continue as usual. If we want to use synthetic division then we have to modify slightly the way in which we deal with the quadratic divisor, as has been described in section 2.6.13. We will show an example of this below. Before we do this we can state the process above as a theorem.

Rational root theorem (alternative version)

Let $f(x)$ be an n^{th} degree polynomial, where $n > 2$, and let p and q be the set of factors of the constant term and the leading term respectively of $f(x)$. Let two possible roots of $f(x)$ be $x = \pm p/q$. Given that $f(x)$ divided by $g(x) = (qx-p)(qx+p)$ gives a remainder $Ax + B$, then r is a root of $f(x) = 0$ if and only if $r = -B/A$.

Proof: By the remainder theorem we have

$$f(x) = (x^2 + bx + c)Q(x) + Ax + B,$$

where $Q(x)$ is the quotient of $f(x)$. If r is a root of the quadratic then $r = x = -B/A$ and we have

$$f(r) = (0)Q(r) + A\left(-\frac{B}{A}\right) + B = 0.$$

So r is a root of $f(x) = 0$.

On the other hand, if r is a root of $f(x)$ we have $f(r) = 0$. Therefore

$$(r^2 + br + c)Q(r) + Ar + B = 0\ .$$

Since r is a roots of the quadratic $r^2 + br + c = 0$, and we obtain

$$Ar + B = 0$$

Therefore $r = -B/A$ as required. ■

The process above also works when $g(x)$ is of the form $ax^2 + bx + c$ (i.e. where the leading coefficient of $g(x)$ is not 1). All we have to be careful of is our use of synthetic division by quadratic divisors. In this case we use the process given in section 2.6.13. Otherwise we can use long division in the normal way.

2.9 A comment about irreducibility: When is it possible to factorise?

All our work has involved polynomials with integer coefficients, and all our work on factorisation has involved factorising polynomials as to get factors with integer coefficient, i.e.

$$x^2 + 3x + 2 = (x + 1)(x + 2).$$

When we talk about the factorisation of a polynomial we usually need to specify what set of coefficients we want to use. Do we want integer coefficients to our factors? Do we want rational coefficients? Real coefficients? Complex coefficients?

For example,

Expression	Factorisable over …			
	… integers	… rationals	… reals	… complex numbers
(1) $x^2 - 1$	Yes $(x+1)(x-1)$	Yes $(x+1)(x-1)$	Yes $(x+1)(x-1)$	Yes $(x+1)(x-1)$
(2) $\frac{3}{2}x^2 - 6$	No	Yes $\left(\frac{3}{2}x-3\right)(x+2)$	Yes $\left(\frac{3}{2}x-3\right)(x+2)$	Yes $\left(\frac{3}{2}x-3\right)(x+2)$
(3) $x^2 - 3$	No	No	Yes $(x-\sqrt{3})(x+\sqrt{3})$	Yes $(x-\sqrt{3})(x+\sqrt{3})$
(4) $x^2 + 1$	No	No	No	Yes $(x-i)(x+i)$

To say that a polynomial is irreducible therefore depends on which set of numbers we are factorising over. For example, the polynomial in (2) cannot be factorised in terms of integer coefficients. Therefore (2) is said to be irreducible over the integers. Similarly (3) is irreducible over the integers and the rationals, and (4) is irreducible over the integers, rationals and reals.

The question now is, Are there any polynomials which are irreducible over the complex numbers? Are there polynomials which cannot be factorised in terms of complex coefficients but only in terms of another set of numbers? The answer is no. The reason for this lies in some advanced maths, but briefly speaking the set of complex numbers is said to be "closed". To understand this consider the following text (I have lost the reference for this, so if you come across this I would appreciate you sending it to me).

4.4 What's missing: algebraic closure

One way to think about the development of number systems is that each system $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ adds the ability to solve equations that have no solutions in the previous system. Some specific examples are

$$\begin{aligned} x+1&=0 && \text{Solvable in } \mathbb{Z} \text{ but not } \mathbb{N} \\ 2x&=1 && \text{Solvable in } \mathbb{Q} \text{ but not } \mathbb{Z} \\ x\cdot x&=2 && \text{Solvable in } \mathbb{R} \text{ but not } \mathbb{Q} \\ x\cdot x+1&=0 && \text{Solvable in } \mathbb{C} \text{ but not } \mathbb{R} \end{aligned}$$

This process stops with the complex numbers $\mathbb{C}$, which consist of pairs of the form $a+bi$ where $i^2=-1$. The reason is that the complex numbers are **algebraically closed**: if you write an equation using only complex numbers, $+$, and $\cdot$, and it has some solution x in any field bigger than $\mathbb{C}$, then x is in $\mathbb{C}$ as well. The down side in comparison to the reals is that we lose order: there is no ordering of complex numbers that satisfies the translation and scaling invariance axioms. As in many other areas of mathematics and computer science, we are forced to make trade-offs based on what is important to us at the time.

Symbol	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{C}$
Name	Naturals	Integers	Rationals	Reals	Complex numbers
Typical element	12	-12	$-\frac{12}{7}$	$\sqrt{12}$	$\sqrt{12}+\frac{22}{7}i$
Associative	Yes	Yes	Yes	Yes	Yes
0 and 1	Yes	Yes	Yes	Yes	Yes
Inverses	No	$+$ only	Yes	Yes	Yes
Associative	Yes	Yes	Yes	Yes	Yes
Ordered	Yes	Yes	Yes	Yes	No
Least upper bounds	Yes	Yes	No	Yes	No
Algebraically closed	No	No	No	No	Yes

Table 4.1: Features of various standard algebras